I. INTRODUCTION TO QUANTUM TRANSPORT IN HIGH LANDAU LEVELS

A. Displacement and momentum transfer

If $\mathbf{k}_i = k_\mathrm{F}(\cos\varphi_i, \sin\varphi_i)$, then $\mathbf{R}_i = R_\mathrm{c}[\cos(\pi/2+\varphi_i), \sin(\pi/2+\varphi_i)] = R_\mathrm{c}(-\sin\varphi_i, \cos\varphi_i)$ and the shift (displacement) of the guiding center in a single scattering act is

$$\Delta\mathbf{R} \equiv \mathbf{R}_2 - \mathbf{R}_1 = R_\mathrm{c}\left(-\sin\varphi_2 + \sin\varphi_1, \cos\varphi_2 - \cos\varphi_i\right) . \tag{1}$$

If $\mathbf{E} = (\mathcal{E}_\mathrm{dc}, 0)$, the energy gained by an electron is

$$\mathcal{W}_{12} \equiv e\mathbf{E}\cdot\Delta\mathbf{R} = e\mathcal{E}_\mathrm{dc}\Delta X = e\mathcal{E}_\mathrm{dc}R_\mathrm{c}\left(\sin\varphi_1 - \sin\varphi_2\right) . \tag{2}$$

Magnitude of the transfered momentum $\Delta\mathbf{k} \equiv \mathbf{k}_2 - \mathbf{k}_1$ is

$$|\Delta\mathbf{k}| = 2k_\mathrm{F}\sin\left|\frac{\Delta\varphi}{2}\right| \le 2k_\mathrm{F} . \tag{3}$$

Since $|\Delta\mathbf{R}|/R_\mathrm{c} = |\Delta\mathbf{k}|/k_\mathrm{F}$, one obtains

$$|\Delta\mathbf{R}| = 2R_\mathrm{c}\sin\left|\frac{\Delta\varphi}{2}\right| \le 2R_\mathrm{c} . \tag{4}$$

B. Transition probabilities and Fermi's Golden rule

Electric current $[\mathbf{E} = (\mathcal{E}_\mathrm{dc}, 0)]$ is

$$j = 2\nu_0 e \int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 \left(\Pi_{x_1\to x_2} - \Pi_{x_2\to x_1}\right) , \tag{5}$$

where the transition probability is given by

$$\Pi_{x_1\to x_2} = \iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2} \iiint d\varepsilon_1 d\varepsilon_2 d\Omega \mathcal{M}_{12}\Gamma_{12}\cdot\delta(x_2 - x_1 - \Delta X)\delta(\varepsilon_2 - \varepsilon_1 - \Omega) . \tag{6}$$

Here, we have introduced Ω which is the *total* energy change during single scattering act,

$$\mathcal{M}_{12} = \tilde{\nu}_1\tilde{\nu}_2 f_1\left[1 - f_2\right] , \tag{7}$$

and Γ_{12} is the scattering rate which depends on the specifics of the problem. Since the system is homogeneous, $\Pi_{x_1\to x_2} = \Pi_{x_2 - x_1} = \Pi_{\Delta X}$ and

$$\int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 \to 2\int_{0}^{\infty} \Delta X d(\Delta X) . \tag{8}$$

Energy conservation removes one integration and we obtain (with $\varepsilon_1 = \varepsilon$)

$$\begin{aligned} f_1 &= f(\varepsilon),\ f_2 = f(\varepsilon+\Omega)\,, \\ \tilde{\nu}_1 &= \tilde{\nu}(\varepsilon),\ \tilde{\nu}_2 = \tilde{\nu}(\varepsilon+\Omega)\,. \end{aligned} \tag{9}$$

We also notice that $f_1[1-f_2] - f_2[1-f_1] = f_1 - f_2$. At $T \gtrsim \Omega$, we can write

$$f_1 - f_2 = f(\varepsilon) - f(\varepsilon+\Omega) \simeq -\partial_\varepsilon f(\varepsilon)\cdot\Omega\,, \tag{10}$$

and the integration over energy reduces to

$$\iint d\varepsilon_1 d\varepsilon_2 \tilde{\nu}_1\tilde{\nu}_2(f_1-f_2)\delta(\varepsilon_2-\varepsilon_1-\Omega) = \Omega\int d\varepsilon[-\partial_\varepsilon f(\varepsilon)]\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\Omega) \equiv \Omega\cdot\langle\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\Omega)\rangle_\varepsilon \tag{11}$$

The equation for current then simplifies to:

$$j = 4\nu_0 e\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\iint d\varepsilon d\Omega\Delta X\cdot\Gamma_{12}\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\Omega)\left[-\partial_\varepsilon f(\varepsilon)\right]\cdot\Omega\,, \tag{12}$$

or, since $\mathcal{W}_{12} = e\mathcal{E}_{\mathrm{dc}}\Delta X$,

$$\mathbf{j}\cdot\mathbf{E} = 4\nu_0\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\iint d\varepsilon d\Omega\mathcal{W}_{12}\cdot\Gamma_{12}\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\Omega)\left[-\partial_\varepsilon f(\varepsilon)\right]\cdot\Omega\,. \tag{13}$$

C. Impurity scattering

For impurity scattering

$$\Gamma_{12} = \tau_{12}^{-1}\delta(\Omega-\mathcal{W}_{12})\,, \tag{14}$$

and Eq. 13 becomes:

$$\mathbf{j}\cdot\mathbf{E} = 4\nu_0\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\frac{\mathcal{W}_{12}^2}{\tau_{12}}\int d\varepsilon\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\mathcal{W}_{12})\left[-\partial_\varepsilon f(\varepsilon)\right] \tag{15}$$

In general, scattering rate depends on the scattering angle $\theta = \varphi_2 - \varphi_1$, $\tau_{12}^{-1} \equiv \tau_{\varphi_1\varphi_2}^{-1} = \tau_\theta^{-1}$.

D. Linear response resistivity and transport scattering rate

At small electric fields we can linearize Eq. 15 by setting $\tilde{\nu}(\varepsilon+\mathcal{W}_{12}) \simeq \tilde{\nu}(\varepsilon)$,

$$\mathbf{j}\cdot\mathbf{E} = 4\nu_0\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\frac{\mathcal{W}_{12}^2}{\tau_{12}}\int d\varepsilon\cdot\tilde{\nu}^2(\varepsilon)\left[-\partial_\varepsilon f(\varepsilon)\right]\,. \tag{16}$$

With $2\varphi_\pm = \varphi_1 \pm \varphi_2$ ($\varphi_1 = \varphi_+ + \varphi_-, \varphi_2 = \varphi_+ - \varphi_-$), the integral over angles,

$$\iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2} \frac{\mathcal{W}_{12}^2}{\tau_{12}} = e^2 \mathcal{E}_{\rm dc}^2 R_{\rm c}^2 \iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2} \frac{(\sin\varphi_1 - \sin\varphi_2)^2}{\tau_{\varphi_1\varphi_2}}, \tag{17}$$

reduces to (Jacobian is equal to 2)

$$\iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2} = 2 \cdot \int_0^{2\pi} \frac{d\varphi_+}{2\pi} \int_0^{\pi} \frac{d\varphi_-}{2\pi}. \tag{18}$$

Using $\sin\varphi_1 - \sin\varphi_2 = 2\cos\varphi_+ \sin\varphi_-$ and $\theta = 2\varphi_-$ we then obtain

$$\left\langle \frac{(\sin\varphi_1 - \sin\varphi_2)^2}{\tau_{\varphi_1\varphi_2}} \right\rangle_{\varphi_1\varphi_2} = 2\cdot 4 \int_0^{2\pi} \frac{d\varphi_+}{2\pi} \cos^2\varphi_+ \int_0^{\pi} \frac{d\varphi_-}{2\pi} \frac{\sin^2\varphi_-}{\tau_{\varphi_-}} = 2\cdot 4\cdot \frac{1}{2}\cdot \int_0^{2\pi} \frac{d(\theta/2)}{2\pi} \frac{1-\cos\theta}{2\tau_\theta}, \tag{19}$$

which defines **transport scattering rate**

$$\frac{1}{\tau_{\rm tr}} \equiv \left\langle \frac{(\sin\varphi_1 - \sin\varphi_2)^2}{\tau_{\varphi_1\varphi_2}} \right\rangle_{\varphi_1\varphi_2} \equiv \left\langle \frac{1-\cos\theta}{\tau_\theta} \right\rangle_\theta \equiv \int \frac{d\theta}{2\pi} \frac{1-\cos\theta}{\tau_\theta} \equiv \frac{1}{\tau_0} - \frac{1}{\tau_1}. \tag{20}$$

With the classical Drude conductivity in magnetic field,

$$\sigma_D = \frac{ne^2\tau_{\rm tr}}{m^*} \cdot \frac{1}{(\omega_{\rm c}\tau_{\rm tr})^2}, \tag{21}$$

the conductivity can be written as:

$$\sigma \equiv \frac{\mathbf{j}\cdot\mathbf{E}}{\mathcal{E}_{\rm dc}^2} = \sigma_D \cdot \int d\varepsilon \tilde{\nu}^2(\varepsilon) \left[-\partial_\varepsilon f(\varepsilon)\right]. \tag{22}$$

E. Shubnikov-de Haas Oscillations

For overlapping Landau levels ($\omega_{\rm c}\tau_0 < 1$), the density of states can be described by its first harmonic,

$$\tilde{\nu}(\varepsilon) = 1 - 2\lambda \cos\left(\frac{2\pi\varepsilon}{\hbar\omega_{\rm c}}\right) = 1 - 2\lambda\cos 2\pi\tilde{\varepsilon}, \tag{23}$$

where

$$\lambda = \exp\left(-\frac{\pi}{\omega_{\rm c}\tau_0}\right) \ll 1 \tag{24}$$

is known as the **Dingle factor**. We then have to evaluate

$$\int d\tilde{\varepsilon} \left(1 - 4\lambda\cos 2\pi\tilde{\varepsilon} + 4\lambda^2 \cos^2 2\pi\tilde{\varepsilon}\right) \left[-\partial_{\tilde{\varepsilon}} f(\tilde{\varepsilon})\right]. \tag{25}$$

With the following notation for energy averaging,

$$\int d\varepsilon\,[-\partial_\varepsilon f(\varepsilon)]\,(\ldots) = \langle\ldots\rangle_\varepsilon\,, \tag{26}$$

we find

$$\begin{aligned} \langle 1\rangle_\varepsilon &= 1\,, \\ \langle\cos 2\pi\tilde{\varepsilon}\rangle_\varepsilon &= \frac{2\pi^2 T/\hbar\omega_{\rm c}}{\sinh 2\pi^2 T/\hbar\omega_{\rm c}}\cos(2\pi\varepsilon_{\rm F}/\hbar\omega_{\rm c})\,, \\ \left\langle\cos^2 2\pi\tilde{\varepsilon}\right\rangle_\varepsilon &\simeq \langle 1/2\rangle_\varepsilon = 1/2\,, \end{aligned} \tag{27}$$

where the last term gives non-oscillatory quantum correction to the conductivity, $\sigma \simeq \sigma_D\left[1+2\lambda^2\right]$. The second term, gives the oscillatory quantum correction to the conductivity (resistivity), which is known as **Shubnikov-de Haas oscillations** (SdHO):

$$\frac{\delta\sigma}{\sigma_D} \simeq \frac{\delta\rho}{\rho_D} \simeq -4\lambda\frac{2\pi^2 T/\hbar\omega_{\rm c}}{\sinh 2\pi^2 T/\hbar\omega_{\rm c}}\cos(2\pi\varepsilon_{\rm F}/\hbar\omega_{\rm c})\,. \tag{28}$$

SdHO can be used to experimentally obtain n_e (period), τ_0 (B-dependence of the amplitude), and m^* (T-dependence of the amplitude).

F. More on energy averaging

In many cases one should calculate energy averaging explicitly:

$$\langle\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\Omega)\rangle_\varepsilon = 1-2\lambda\left[\langle\cos 2\pi\tilde{\varepsilon}\rangle_\varepsilon + \left\langle\cos 2\pi(\tilde{\varepsilon}+\tilde{\Omega})\right\rangle_\varepsilon\right] + 4\lambda^2\left\langle\cos 2\pi\tilde{\varepsilon}\cos 2\pi(\tilde{\varepsilon}+\tilde{\Omega})\right\rangle_\varepsilon \tag{29}$$

At high T, linear in λ term is exponentially suppressed (cf. SdHO), but the quadratic term survives:

$$\left\langle\cos 2\pi\tilde{\varepsilon}\cos 2\pi(\tilde{\varepsilon}+\tilde{\Omega})\right\rangle_\varepsilon = \cos 2\pi\tilde{\Omega}\left\langle\cos^2 2\pi\tilde{\varepsilon}\right\rangle_\varepsilon - \sin 2\pi\tilde{\Omega}\,\langle\cos 2\pi\tilde{\varepsilon}\sin 2\pi\tilde{\varepsilon}\rangle_\varepsilon = \frac{\cos 2\pi\tilde{\Omega}}{2} \tag{30}$$

G. Nonlinear response resistivity: Hall field-induced resistance oscillations

Non-linear effects originate from the correction quadratic in λ. From Eq. 15 we have

$$\mathbf{j}\cdot\mathbf{E} = 4\nu_0\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\frac{\mathcal{W}_{12}^2}{\tau_{12}}\,\langle\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon+\mathcal{W}_{12})\rangle_\varepsilon \tag{31}$$

and with $w_{12} = \mathcal{W}_{12}/\hbar\omega_c = (\epsilon_{dc}/2)(\sin\varphi_1 - \sin\varphi_2)$, where $\epsilon_{dc} \equiv e\mathcal{E}_{dc}(2R_c)/\hbar\omega_c$ the relevant term can be written as

$$\delta(\mathbf{j}\cdot\mathbf{E}) = 4\nu_0 \cdot (2\lambda)^2(\hbar\omega_c)^2 \cdot \iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\frac{w_{12}^2}{\tau_{12}} \cdot \frac{\cos 2\pi w_{12}}{2} \tag{32}$$

or

$$\delta\sigma = \frac{\delta(\mathbf{j}\cdot\mathbf{E})}{\mathcal{E}_{dc}^2} = 2\nu_0(2\lambda)^2 e^2 R_c^2 \iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\frac{(\sin\varphi_1 - \sin\varphi_2)^2}{\tau_{\varphi_1\varphi_2}} \cdot \cos\left[\pi\epsilon_{dc}(\sin\varphi_1 - \sin\varphi_2)\right] . \tag{33}$$

At large electric fields, $\pi\epsilon_{dc} \gg 1$, the integral can be evaluated using a stationary phase approximation. Indeed, the integral will be close to zero unless $\sin\varphi_1 - \sin\varphi_2 = \text{const}$ and the main contribution will be from angles such that $\sin\varphi_1 - \sin\varphi_2 \simeq \pm 2$ or, equivalently, $\varphi_1 \simeq \pm\pi/2$ and $\varphi_2 \simeq \mp\pi/2$. This corresponds to electron backscattering, $\theta = \pm\pi$ and the relevant scattering rate is $1/\tau_\pi$. We note that

$$\cos\left[\pi\epsilon_{dc}(\sin\varphi_1 - \sin\varphi_2)\right] = \Re\left\{e^{i\pi\epsilon_{dc}\sin\varphi_1} e^{-i\pi\epsilon_{dc}\sin\varphi_2}\right\} , \tag{34}$$

and

$$\frac{(\sin\varphi_1 - \sin\varphi_2)^2}{\tau_{\varphi_1\varphi_2}} \simeq \frac{4}{\tau_\pi} . \tag{35}$$

We then expand the phase near the stationary point,

$$\sin\varphi_1 \simeq 1 - \varphi_1^2/2, \sin\varphi_2 \simeq -1 + \varphi_2^2/2 , \tag{36}$$

and obtain

$$e^{i2\pi\epsilon_{dc}} \iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2} e^{-i\pi\epsilon_{dc}\varphi_1^2/2} e^{-i\pi\epsilon_{dc}\varphi_2^2/2} , \tag{37}$$

$$\int \frac{d\varphi}{2\pi} e^{-i\pi\epsilon_{dc}\varphi^2/2} = \frac{1}{2\pi}\sqrt{\frac{\pi}{\pi\epsilon_{dc}/2}} e^{-i\pi/4} = \frac{1}{\pi}\frac{e^{-i\pi/4}}{\sqrt{2\epsilon_{dc}}} , \tag{38}$$

and

$$\iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2}(...) \simeq \frac{4}{\tau_\pi}\frac{1}{\pi^2}\frac{1}{2\epsilon_{dc}}\Re\left\{e^{i2\pi\epsilon_{dc} - i\pi/2}\right\} = \frac{2}{\pi\tau_\pi}\frac{\sin 2\pi\epsilon_{dc}}{\pi\epsilon_{dc}} . \tag{39}$$

This result should be doubled because there are two stationary points. We then obtain

$$\delta(\mathbf{j}\cdot\mathbf{E}) = 8\nu_0(2\lambda)^2 e^2\mathcal{E}_{dc}^2 R_c^2 \frac{1}{\pi\tau_\pi}\frac{\sin 2\pi\epsilon_{dc}}{\pi\epsilon_{dc}} \tag{40}$$

and

$$\delta\sigma = 8\nu_0(2\lambda^2)e^2 R_c^2 \frac{1}{\pi\tau_\pi}\frac{\sin 2\pi\epsilon_{dc}}{\pi\epsilon_{dc}} = \sigma_D \cdot 2\frac{(2\lambda)^2}{\pi}\frac{\tau_{tr}}{\tau_\pi}\frac{\sin 2\pi\epsilon_{dc}}{\pi\epsilon_{dc}} . \tag{41}$$

In classically strong magnetic fields, one has

$$\frac{\delta\rho}{\rho_D} \simeq \frac{\delta\sigma}{\sigma_D} = 2\frac{(2\lambda)^2}{\pi}\frac{\tau_{tr}}{\tau_\pi}\frac{\sin 2\pi\epsilon_{dc}}{\pi\epsilon_{dc}}\,. \tag{42}$$

Experiments measure differential resistivity:

$$r \equiv \frac{dV}{dI} = \frac{d(\rho j)}{dj} = \frac{d(\epsilon_{dc}\rho)}{d\epsilon_{dc}} = \rho_D + \frac{d(\epsilon_{dc}\delta\rho)}{d\epsilon_{dc}} = \rho_D + \frac{(4\lambda)^2}{\pi}\frac{\tau_{tr}}{\tau_\pi}\cos 2\pi\epsilon_{dc}\,. \tag{43}$$

The second term in this expression describes **Hall field-induced resistance oscillations**. Note that the amplitude does not decay with ϵ_{dc} at a fixed magnetic field.

H. Transport in irradiated 2DEG

When a 2DES is exposed to monochromatic radiation with frequency ω one can expect absorption and emission of photons. In the regime linear in microwave power $\mathcal{P}_\omega$ (this is oversimplified, see below), the scattering rate is

$$\Gamma_{12} = \frac{\mathcal{P}_\omega}{\tau_{12}}\delta(\Omega \pm \hbar\omega - \mathcal{W}_{12})\,. \tag{44}$$

From Eq. 15

$$\mathbf{j}\cdot\mathbf{E} = 4\nu_0 \iint \frac{d\varphi_1 d\varphi_2}{(2\pi)^2}\frac{\mathcal{W}_{12}\,(\mathcal{W}_{12} \pm \hbar\omega)}{\tau_{12}}\,\langle\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon + \mathcal{W}_{12} \pm \hbar\omega)\rangle_\varepsilon\ , \tag{45}$$

where, with $w_{12} = \mathcal{W}_{12}/\hbar\omega_c$, $\epsilon_{ac} = \omega/\omega_c$, one obtains

$$\langle\tilde{\nu}(\varepsilon)\tilde{\nu}(\varepsilon + \mathcal{W}_{12} \pm \hbar\omega)\rangle_\varepsilon = \frac{\cos[2\pi(w_{12} \pm \epsilon_{ac})]}{2} = \frac{\cos 2\pi w_{12}\cos 2\pi\epsilon_{ac} \mp \sin 2\pi w_{12}\sin 2\pi\epsilon_{ac}}{2}\,. \tag{46}$$

The dominant contribution again will be quadratic in λ,

$$\delta(\mathbf{j}\cdot\mathbf{E}) = 4\nu_0 \cdot (2\lambda)^2 \cdot \frac{1}{2} \cdot (\hbar\omega_c)^2 \cdot \langle\ldots\rangle_{\varphi_1\varphi_2}\ , \tag{47}$$

where

$$\langle\ldots\rangle \equiv \left\langle \frac{w_{12}(w_{12} \pm \epsilon_{ac})(\cos 2\pi w_{12}\cos 2\pi\epsilon_{ac} \mp \sin 2\pi w_{12}\sin 2\pi\epsilon_{ac})}{\tau_{12}} \right\rangle = \langle 1\rangle + \langle 2\rangle + \langle 3\rangle + \langle 4\rangle\ . \tag{48}$$

Notice that terms $\langle 2\rangle$ and $\langle 3\rangle$ vanish because they contain odd functions:

$$\begin{aligned}
\langle 1\rangle &= \cos 2\pi\epsilon_{ac}\left\langle \frac{w_{12}^2\cos 2\pi w_{12}}{\tau_{12}}\right\rangle ,\\
\langle 2\rangle &= \pm\epsilon_{ac}\cos 2\pi\epsilon_{ac}\left\langle \frac{w_{12}\cos 2\pi w_{12}}{\tau_{12}}\right\rangle = 0\,,\\
\langle 3\rangle &= \mp\sin 2\pi\epsilon_{ac}\left\langle \frac{w_{12}^2\sin 2\pi w_{12}}{\tau_{12}}\right\rangle = 0\,,\\
\langle 4\rangle &= -\epsilon_{ac}\sin 2\pi\epsilon_{ac}\left\langle \frac{w_{12}\sin 2\pi w_{12}}{\tau_{12}}\right\rangle .
\end{aligned}$$

Therefore we have:

$$\langle \ldots \rangle_{\varphi_1\varphi_2} = \cos 2\pi\epsilon_{\rm ac} \left\langle \frac{w_{12}^2 \cos 2\pi w_{12}}{\tau_{12}} \right\rangle - \epsilon_{\rm ac} \sin 2\pi\epsilon_{\rm ac} \left\langle \frac{w_{12} \sin 2\pi w_{12}}{\tau_{12}} \right\rangle$$

I. Linear response resistivity in irradiated 2DEG: Microwave-induced resistance oscillations

At small electric fields, $w_{12} \ll 1$, we have

$$\langle \ldots \rangle_{\varphi_1\varphi_2} \simeq \left\langle \frac{w_{12}^2}{\tau_{12}} \right\rangle_{\varphi_1\varphi_2} \cdot (\cos 2\pi\epsilon_{\rm ac} - 2\pi\epsilon_{\rm ac} \sin 2\pi\epsilon_{\rm ac})$$

and one obtains (simplified) expression for **microwave-induced resistance oscillations**:

$$\delta\sigma = \sigma_D \cdot 2\lambda^2 \mathcal{P}_\omega (\cos 2\pi\epsilon_{\rm ac} - 2\pi\epsilon_{\rm ac} \sin 2\pi\epsilon_{\rm ac}) \simeq -\sigma_D \cdot 2\lambda^2 \mathcal{P}_\omega \cdot 2\pi\epsilon_{\rm ac} \sin 2\pi\epsilon_{\rm ac} \,, \tag{49}$$

where the last approximation is valid because $\epsilon_{\rm ac} \gtrsim 1$.

The correct expression for the microwave-induced disorder scattering rate is:

$$\Gamma_{12} = \frac{\mathcal{P}_\omega}{\tau_{12}} \cdot \sin^2 \frac{\varphi_1 - \varphi_2}{2} \left[\sum_{\pm} \frac{1}{2} \delta(\Omega \pm \hbar\omega - \mathcal{W}_{12}) - \delta(\Omega - \mathcal{W}_{12}) \right] . \tag{50}$$

One therefore obtains:

$$\begin{aligned} \left\langle \frac{(\sin\varphi_1 - \sin\varphi_2)^2}{\tau_{12}} \sin \frac{\varphi_1 - \varphi_2}{2} \right\rangle &= 2 \cdot 4 \int \frac{d\varphi_+}{2\pi} \cos^2 \varphi_+ \int \frac{d\varphi_-}{2\pi} \frac{\sin^4 \varphi_-}{\tau_{\varphi_-}} \\ &= 2 \cdot 4 \cdot \frac{1}{2} \cdot \int \frac{d(\theta/2)}{2\pi} \frac{(1-\cos\theta)^2}{4\tau_\theta} \equiv \frac{1}{4\tau^\star} \,, \end{aligned} \tag{51}$$

where

$$\frac{1}{\tau^\star} \equiv 2 \left\langle \frac{(1-\cos\theta)^2}{\tau_\theta} \right\rangle_\theta \equiv \frac{3}{\tau_0} - \frac{4}{\tau_1} + \frac{1}{\tau_2} \tag{52}$$

Indeed

$$\begin{aligned} \left\langle (1-\cos\theta)^2 \cdot \frac{1}{\tau_0} \right\rangle_\theta &= \frac{1}{\tau_0} \left[1 + \left\langle \cos^2\theta \right\rangle \right] = \frac{3}{2\tau_0} \,, \\ \left\langle (1-\cos\theta)^2 \cdot \frac{2\cos\theta}{\tau_1} \right\rangle_\theta &= \frac{2}{\tau_1} \left\langle -2\cos^2\theta \right\rangle = -\frac{2}{\tau_1} \,, \\ \left\langle (1-\cos\theta)^2 \cdot \frac{2\cos 2\theta}{\tau_2} \right\rangle_\theta &= \frac{2}{\tau_2} \left\langle \cos^2\theta \cos 2\theta \right\rangle = \frac{1}{2\tau_2} \,, \\ \left\langle (1-\cos\theta)^2 \cdot \frac{2\cos n\theta}{\tau_n} \right\rangle_\theta &= 0, n > 2 \,. \end{aligned} \tag{53}$$

The expression for MIRO (displacement contribution) is then given by

$$\begin{aligned}\delta\rho &= \rho_D \cdot 2\lambda^2 \mathcal{P}_\omega \frac{\tau_{\rm tr}}{4\tau^\star}\left(\cos 2\pi\epsilon_{\rm ac} - 2\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm ac} - 1\right) \\ &= -\rho_D \cdot 2\lambda^2 \mathcal{P}_\omega \frac{\tau_{\rm tr}}{2\tau^\star}\left(\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm ac} + \sin^2\pi\epsilon_{\rm ac}\right) \approx -\rho_D \cdot 2\lambda^2 \mathcal{P}_\omega \frac{\tau_{\rm tr}}{2\tau^\star}\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm ac}\,. \end{aligned} \quad (54)$$

J. Non-linear response of irradiated 2DEG

At strong dc fields we will have an oscillatory correction which is controlled by both $\epsilon_{\rm ac}$ and $\epsilon_{\rm dc}$. We need to evaluate the following averages:

$$\left\langle \frac{w_{12}^2\cos 2\pi w_{12}}{\tau_{12}}\sin^2\frac{\varphi_1-\varphi_2}{2}\right\rangle, \quad \left\langle \frac{w_{12}\sin 2\pi w_{12}}{\tau_{12}}\sin^2\frac{\varphi_1-\varphi_2}{2}\right\rangle.$$

One can see that the presence of $\sin^2\varphi_-$ will not affect our result since its value at the stationary points $\varphi_\mp = \pm\pi/2$ is equal to unity. Therefore we can use our previous result (2 appears because there are two stationary points):

$$\left\langle \frac{w_{12}^2\cos 2\pi w_{12}}{\tau_{12}}\right\rangle = 2\cdot\frac{4(\epsilon_{\rm dc}/2)^2}{\pi\tau_\pi}\frac{\sin 2\pi\epsilon_{\rm dc}}{\pi\epsilon_{\rm dc}} = 2\cdot\frac{\epsilon_{\rm dc}\sin 2\pi\epsilon_{\rm dc}}{\pi^2\tau_\pi}\,.$$

Similarly we can show that

$$\left\langle \frac{w_{12}\sin 2\pi w_{12}}{\tau_{12}}\right\rangle = -2\cdot\frac{2(\epsilon_{\rm dc}/2)}{\pi\tau_\pi}\frac{\cos 2\pi\epsilon_{\rm dc}}{\pi\epsilon_{\rm dc}} = -2\cdot\frac{\cos 2\pi\epsilon_{\rm dc}}{\pi^2\tau_\pi}\,.$$

Indeed,

$$\sin\left[\pi\epsilon_{\rm dc}(\sin\varphi_1 - \sin\varphi_2)\right] = \Im\left\{e^{i\pi\epsilon_{\rm dc}\sin\varphi_1}e^{-i\pi\epsilon_{\rm dc}\sin\varphi_2}\right\}, \quad (55)$$

and for $\varphi_1 = \pi/2$, $\varphi_2 = -\pi/2$,

$$\frac{(\sin\varphi_1 - \sin\varphi_2)}{\tau_{\varphi_1\varphi_2}} \simeq \frac{2}{\tau_\pi}\,. \quad (56)$$

We then expand the phase, $\sin\varphi_1 \simeq 1-\varphi_1^2/2$, $\sin\varphi_2 \simeq -1+\varphi_2^2/2$ and obtain

$$e^{i2\pi\epsilon_{\rm dc}}\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}e^{-i\pi\epsilon_{\rm dc}\varphi_1^2/2}e^{-i\pi\epsilon_{\rm dc}\varphi_2^2/2}\,, \quad (57)$$

$$\int\frac{d\varphi}{2\pi}e^{-i\pi\epsilon_{\rm dc}\varphi^2/2} = \frac{1}{2\pi}\sqrt{\frac{\pi}{\pi\epsilon_{\rm dc}/2}}e^{-i\pi/4} = \frac{1}{\pi}\frac{e^{-i\pi/4}}{\sqrt{2\epsilon_{\rm dc}}}\,, \quad (58)$$

and

$$\iint\frac{d\varphi_1 d\varphi_2}{(2\pi)^2}(\ldots) \simeq \frac{2}{\tau_\pi}\frac{1}{\pi^2}\frac{1}{2\epsilon_{\rm dc}}\Im\left\{e^{i2\pi\epsilon_{\rm dc}-i\pi/2}\right\} = -\frac{2}{\pi\tau_\pi}\frac{\cos 2\pi\epsilon_{\rm dc}}{\pi\epsilon_{\rm dc}}\,. \quad (59)$$

Correction to conductivity:

$$\frac{\delta\sigma}{\sigma_D} = \frac{(4\lambda)^2\mathcal{P}_\omega\tau_{\rm tr}}{\pi^2\epsilon_{\rm dc}^2\tau_\pi}\left(\epsilon_{\rm dc}\cos 2\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm dc} + \epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm ac}\cos 2\pi\epsilon_{\rm dc}\right) . \tag{60}$$

Correction to differential resistivity:

$$r \equiv \frac{dV}{dI} = \frac{d(\rho j)}{dj} = \frac{d(\epsilon_{\rm dc}\rho)}{d\epsilon_{\rm dc}} = \rho_D + \frac{d(\epsilon_{\rm dc}\delta\rho)}{d\epsilon_{\rm dc}} , \tag{61}$$

where

$$\frac{d}{d\epsilon_{\rm dc}}\left(\cos 2\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm dc} + \frac{\epsilon_{\rm ac}}{\epsilon_{\rm dc}}\sin 2\pi\epsilon_{\rm ac}\cos 2\pi\epsilon_{\rm dc}\right) \simeq 2\pi\left(\cos 2\pi\epsilon_{\rm ac}\cos 2\pi\epsilon_{\rm dc} - \frac{\epsilon_{\rm ac}}{\epsilon_{\rm dc}}\sin 2\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm dc}\right) .$$

As a result, one has

$$\frac{\delta r}{\rho_D} = \frac{(4\lambda)^2 2\mathcal{P}_\omega}{\pi}\frac{\tau_{\rm tr}}{\tau_\pi}\frac{1}{\epsilon_{\rm dc}}\left(\epsilon_{\rm dc}\cos 2\pi\epsilon_{\rm ac}\cos 2\pi\epsilon_{\rm dc} - \epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm dc}\right) , \tag{62}$$

which can be re-written in a symmetric form which has transparent physical meaning. With $\varepsilon_\pm = \epsilon_{\rm ac} \pm \epsilon_{\rm dc}$,

$$\frac{\delta r^{(1)}}{\rho_D} = \frac{(4\lambda)^2\mathcal{P}_\omega}{\pi}\frac{\tau_{\rm tr}}{\tau_\pi}\frac{1}{\epsilon_{\rm dc}}\cdot\left(\varepsilon_+\cos 2\pi\varepsilon_+ - \varepsilon_-\cos 2\pi\varepsilon_-\right)$$

In addition, there will be a term not containing $\epsilon_{\rm ac}$ which can be obtained from the above expression by setting $\epsilon_{\rm ac} = 0$,

$$\frac{\delta r^{(2)}}{\rho_D} = -\frac{(4\lambda)^2 2\mathcal{P}_\omega}{\pi}\frac{\tau_{\rm tr}}{\tau_\pi}\cos 2\pi\epsilon_{\rm dc} \tag{63}$$

and a nonlinear term obtained earlier in the absence of radiation:

$$\frac{\delta r^{(3)}}{\rho_D} = \frac{(4\lambda)^2}{\pi}\frac{\tau_{\rm tr}}{\tau_\pi}\cos 2\pi\epsilon_{\rm dc} \tag{64}$$

The total correction oscillatory correction to r is then given by

$$\frac{\delta r}{\rho_D} = \frac{(4\lambda)^2}{\pi}\frac{\tau_{\rm tr}}{\tau_\pi}\left\{(1 - 2\mathcal{P}_\omega)\cos 2\pi\epsilon_{\rm dc} + 2\mathcal{P}_\omega\left(\cos 2\pi\epsilon_{\rm ac}\cos 2\pi\epsilon_{\rm dc} - \frac{\epsilon_{\rm ac}}{\epsilon_{\rm dc}}\sin 2\pi\epsilon_{\rm ac}\sin 2\pi\epsilon_{\rm dc}\right)\right\}. \tag{65}$$