Review

- Angular variables
  - q = L / r
  - w = v / r

Chap. 9 sec.1

- Moment of Inertia
  - I =S $m_i r_i^2$
  - I = M r^2 (ring)
  - I = f M r^2

Chap. 9 sec 2, 3

- Rolling without slipping
  - v_{com} = w r

Chap. 9 sec 4, 6

- Rotational Kinetic Energy
  - $KE_{rotation}$ = (1/2) I w^2

Chap. 9 sec 5

Preview of Dynamics

Rotations Described by Acceleration

Rotational Kinematics — Chap. 9 sec.1

Radial acceleration

Tangential acceleration

Angular acceleration

Forces give accelerations

Rotational Dynamics

Torque — Chap. 9 sec. 2-4, 6

Moments of inertia

Vector nature of rotations

Directions — Chap. 10 sec 1

Rotations

From the point of view of Forces (dynamics)

Study the interactions of realistic objects

First step, describe the motion

A simple case - two parts of a rotating disk

Our old formulation of kinematics is correct but complicated.

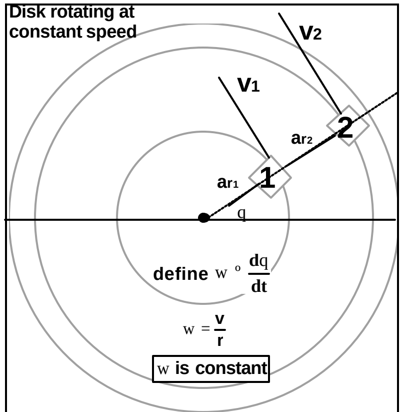
Velocity & acceleration different for each part

Make a simpler description of rotational motion

2 different cases:

1. Uniform Circular Motion
   all acceleration is radial
2. Disk speeding up or slowing down
   acceleration is not radial
   radial component
   tangential component

Disk rotating at constant speed



Time for object 1 to go around by q
equals
Time for object 2 to go around by q

w2 equal w1

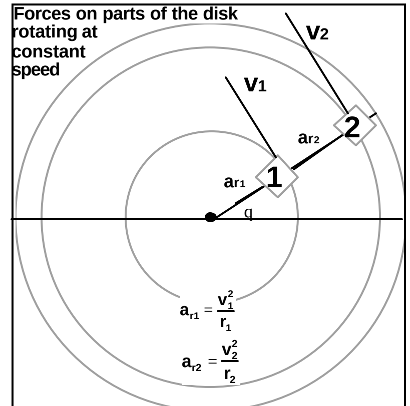
What is the relationship between
v2 and v1   ar2 and ar1

Kinematics

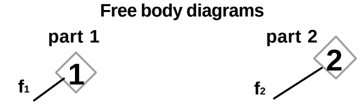
For an object rotating at a constant speed

Old formulation	New formulation
Velocity	Angular velocity
$\vec{v}$ is NOT constant	$\vec{w}$ is constant
v is constant	w is constant
$\bar{\vec{v}} = \frac{\vec{r}_f - \vec{r}_i}{Dt}$	$\bar{\vec{w}} = \frac{\vec{q}_f - \vec{q}_i}{Dt}$
v is NOT $\frac{r_f - r_i}{Dt}$	w is $\frac{q_f - q_i}{Dt}$
$\vec{v} = \frac{d\vec{r}}{dt}$	$\vec{w} = \frac{d\vec{q}}{dt}$
Acceleration	Angular acceleration
$\vec{a} = \frac{d\vec{v}}{dt}$	$\vec{a} = \frac{d\vec{w}}{dt}$
$a = \frac{v^2}{r}$ (radial)	a = 0

Forces on parts of the disk rotating at constant speed



Free body diagrams



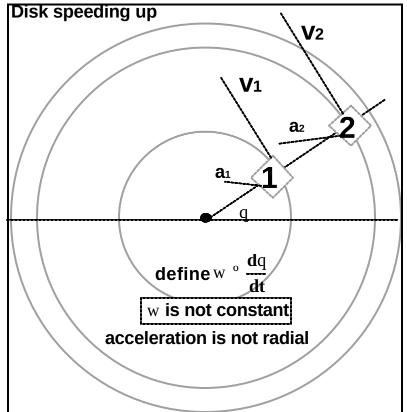
Rotating Object Increasing Angular Speed

Old formulation	New formulation
Velocity	Angular velocity
v is NOT constant	w is NOT constant
v is NOT $\frac{r_f - r_i}{Dt}$	w is NOT $\frac{q_f - q_i}{Dt}$
$\vec{v} = \frac{d\vec{r}}{dt}$	$\vec{w} = \frac{d\vec{q}}{dt}$
Acceleration	Angular acceleration
$\vec{a}$ is NOT constant	$\vec{a}$ is constant
a is NOT $\frac{v_f - v_i}{Dt}$	$a = \frac{w_f - w_i}{Dt}$
$\vec{a} = \frac{d\vec{v}}{dt}$	$\vec{a} = \frac{d\vec{w}}{dt}$

$a_t = \frac{dv}{dt}$

$a_r = \frac{v^2}{r}$ $\qquad a^2 = a_t^2 + a_r^2$

Disk speeding up

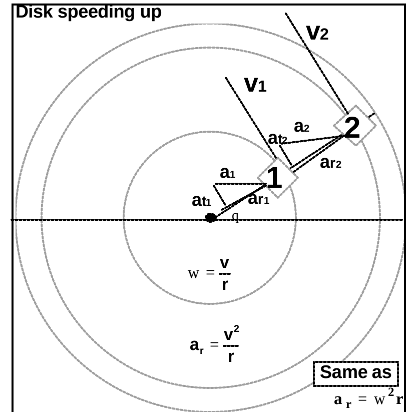


Time for object 1 to go around by q
equals
Time for object 2 to go around by q

v_2 greater than v_1 $\qquad a_2$ greater than a_1

w_2 equal w_1 $\qquad a_2$ equal a_1

Disk speeding up



Tangential acceleration is the rate of change of the magnitude of the instantaneous velocity

$v = rw$

$\frac{dv}{dt} = r\frac{dw}{dt}$ $\qquad a_t = r\frac{dw}{dt}$

The tangential acceleration depends on the position of the part of the disk

$a_{1t} = r_1\frac{dw}{dt}$ $\qquad a_{2t} = r_2\frac{dw}{dt}$

a_{2t} is greater than a_{1t}

$\frac{dw}{dt}$ is the same for all parts

define $a \equiv \frac{dw}{dt}$

a is called the angular acceleration

$ra = a_t$

For Circular Motion - $\vec{v} \wedge \vec{r}$

$w = \frac{dq}{dt}$ $\qquad v = rw$ $\qquad a_r = \frac{v^2}{r}$

$a = \frac{dw}{dt}$ $\qquad ra = a_t$ $\qquad \vec{a} = a_r\hat{r} + a_t\hat{t}$

Only radial components of force cause radial acceleration

Only tangential components of force cause tangential acceleration

---

Constant Angular Acceleration

Special Case: a is constant, $a = \bar{a}$

Find q in terms of a and t

$a = \frac{dw}{dt}$ $\qquad w = \frac{dq}{dt}$

$a = \frac{d}{dt}\left(\frac{dq}{dt}\right)$

What function of time (q) gives a constant (a) when you take two time derivatives?

$q = \frac{1}{2}at^2 + Bt + C$

Find constants B and C

pick a time

at t = 0, $q = q_o$

$q_o = C$

$q = \frac{1}{2}at^2 + Bt + q_o$

To find B, take the derivative with respect to time

$\frac{dq}{dt} = at + B$

$w = at + B$

pick a time

at t = 0, $w = w_o$

$w_o = B$

$q = \frac{1}{2}at^2 + w_o t + q_o$

Dynamics

In order to keep an object rotating

constant w

a = 0

another object must apply a radial force

In order to change an object's rotation

- Start something rotating
- Rotate faster
- Rotate slower
- Stop something rotating

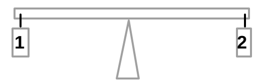
change w

a Not = 0

another object must apply a tangential force as well as a radial force

Now we look in more detail at changing an object's rotation

A Simple System

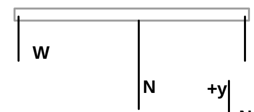


Bar: massless

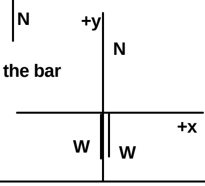
Objects (1 & 2): equal weights

The bar is balanced
nothing moves

Free body diagram of the bar

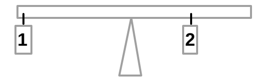


Force diagram of the bar



N - W - W = ma = 0

Move one of the weights



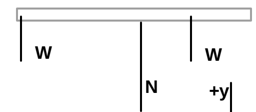
Bar: massless

Objects (1 & 2): equal weights

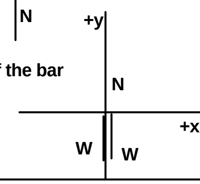
Is the bar is balanced?
no

Do the forces predict the outcome?

Free body diagram of the bar



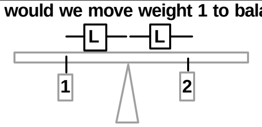
Force diagram of the bar



N - W - W = ma = 0

Need more than the forces

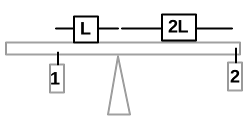
Where would we move weight 1 to balance?



Both the force and the distance from the pivot point are important.

Suppose weight of 1 is twice weight of 2.

Where would you move 2 to balance?



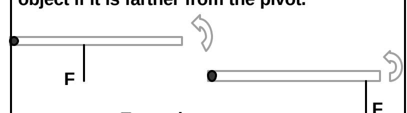
The force times the distance to the pivot is the important quantity

Only the tangential component of the force

Define t = r F_t

t is called the torque

A given tangential force is more effective in changing the rotational motion of an object if it is farther from the pivot.



Examples:
- Doors
- Wrenches

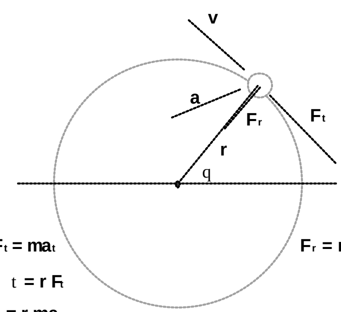
A component of force toward the pivot point will not change the objects rotational motion

A coordinate axis through the pivot point perpendicular to the plane of rotation is called The axis of rotation

How is the torque related to the angular acceleration?

Simple Example

One object attached by a massless rod to a pivot point pushed perpendicular to the rod



$F_t = ma_t$

$F_r = ma_r$

t = r F_t

t = r ma_t

a_t = r a

t = r mr a

t = mr^2 a

In this simple situation, I = mr^2

t = I a

Review

Define a quantity called torque

$\tau = r F_t$

distance to pivot point — tangential component of force (perpendicular to r)

Use Newton's 2nd Law

$\Sigma F_t = m a$

Use definition of angular acceleration

$a_t = r\alpha$

Get

$\tau = mr^2\alpha$

Just another way of saying

Newton's 2nd Law

For many forces just add all the torques and all the mr^2

$\Sigma\overline{\tau} = (\Sigma m_i r_i^2)\overline{\alpha} = I\overline{\alpha}$

Direction of Angular Velocity

To add torques need to know their directions

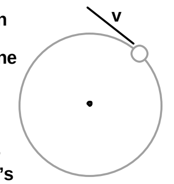
Define a direction for α

Define a direction for ω

Only consistent direction for angular velocity is perpendicular to the plane of the circle

Right hand rule for direction of ω

Curl the fingers of your right hand in the object's direction of motion.



A top view

Your thumb points in the direction of ω

In this case ω is out of the page.

From a side view

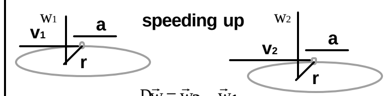
angular velocity is up

angular velocity is down

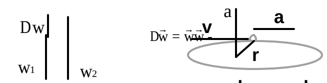
Direction of Angular Acceleration

use definition $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t}$

speeding up



$\Delta\vec{\omega} = \vec{\omega}_2 - \vec{\omega}_1$



angular acceleration is up

Draw a situation where angular acceleration is down.

Right hand rule

Curl the fingers of right hand in the direction the object's tangential acceleration.

Your thumb points in the direction of α

Torque

$$\Sigma\vec{\tau} = \left(\Sigma_i m_i r_i^2\right)\vec{\alpha}$$

Direction of the SUM of all the torque exerted by other objects gives

the direction of the angular acceleration of the object of interest

The magnitude of the angular acceleration of an object depends on

The torque exerted on it by other objects

AND

The distribution of the mass of the object

Moment of inertia $\left(\Sigma_i m_i r_i^2\right) \equiv I$

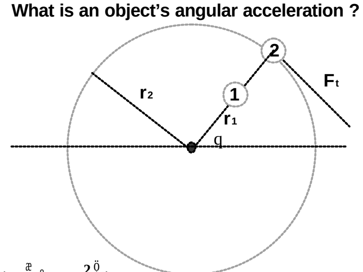
$\Sigma\vec{\tau} = I\vec{\alpha}$ Another way of writing Newton's second law

$\Sigma\vec{F} = m\vec{a}$

Another Simple Example

Two objects are attached at different positions along a massless rod that rotates about a pivot point. The outer object is pushed perpendicular to the rod.

What is an object's angular acceleration ?



$$\Sigma\vec{\tau} = \left(\Sigma_i m_i r_i^2\right)\vec{\alpha}$$

$$r_2F_t = \left(m_1r_1^2 + m_2r_2^2\right)\alpha$$

$$\frac{r_2F_t}{\left(m_1r_1^2 + m_2r_2^2\right)} = \alpha$$

Approaches to Solving Problems

Kinematics

$\vec{v} = \frac{d\vec{r}}{dt}$ $\quad \vec{\omega} = \frac{d\vec{\theta}}{dt}$ $\quad \omega = \frac{v}{r}$ $\quad a_r = \frac{v^2}{r}$

$\vec{a} = \frac{d\vec{v}}{dt}$ $\quad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$ $\quad \alpha = \frac{a_t}{r}$

Dynamics

$\Sigma\vec{F} = m\vec{a}$ $\quad \Sigma\vec{\tau} = I\vec{\alpha}$

$\vec{\tau} = \vec{r}\times\vec{F}$ $\quad I = \Sigma m_i r_i^2 = \int r^2 dm$

Conservation

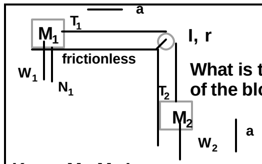
$E_f - E_i = E_{transfer}$ $\quad E_{transfer} = \int\vec{F}\cdot d\vec{\ell}$

$\vec{p}_f - \vec{p}_i = \vec{p}_{transfer}$ $\quad \vec{p}_{transfer} = \int\vec{F}dt$

Rotational Dynamics Example

Using a real pulley (not massless)

You wish to safely lower a pallet of bricks down the side of a building. To do this you attach the pallet to a rope that goes over a pulley with a moment of inertia and radius that you know. After going over the pulley, the rope goes horizontally across the top of the roof and is attached to a box of concrete blocks that slides on a plastic base. To make sure this operation is safe, you calculate the maximum possible acceleration of the pallet of bricks which occurs if the frictional force between the box and the roof is negligible. You will later measure the mass of the box of concrete blocks and the pallet of bricks.



I, r

What is the acceleration of the block?

Know M_1, M_2, I, r

Approach:

Get acceleration using Newton's 2nd law on each block independently.

$$\sum \vec{F} = m\vec{a}$$

Get the relationship between T_1 and T_2 from the torque on the pulley using Newton's 2nd law

$$\sum \vec{\tau} = m\vec{a}$$

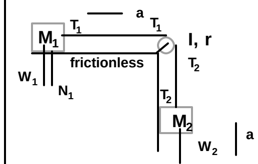
Get angular acceleration of pulley from the tangential acceleration of its edge. Tangential acceleration of edge of pulley is the same as the acceleration of rope.

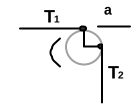
assume rope does not slip on pulley

Assume massless rope

Does $T_1 = T_2$?

To find out examine the motion of the pulley





Pulley's rotation is increasing

a is not zero

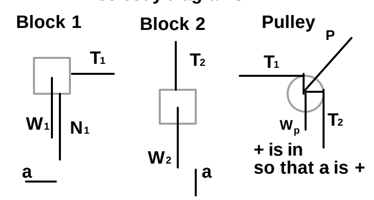
$$a = \frac{a}{r}$$

St = I a

St = T_2 r - T_1 r

T_1 ? T_2 Because St is NOT zero.

Free body diagrams



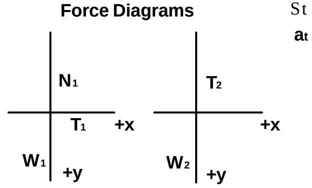
+ is in so that a is +

St = T_2 r - T_1 r

St = Ia

a_t = r a

Force Diagrams



W = Mg

$T_1 = M_1a$

$N_1 - M_1g = 0$

$M_2g - T_2 = M_2a$

Target: a

Find	Equation	#	unknowns
Find a	$T_1 = M_1a$	1	a, T_1
Find T_1	St = T_2 r - T_1 r	2	St, T_2
Find St	St = Ia	3	a
Find a	a = r a	4	
Find T_2	$M_2g - T_2 = M_2a$	5	

5 unknowns, 5 equations

---

5 $M_2g - M_2a = T_2$ Into [2]

St = $(M_2g - M_2a)$ r - T_1 r

Solve [4] for a and put into [3]

St = $\frac{Ia}{r}$ Put into [2]

$$I\frac{a}{r} = (M_2g - M_2a)r - T_1r$$

$$I\frac{a}{r^2} - (M_2g - M_2a) = -T_1 \quad \text{Into [1]}$$

$$I\frac{a}{r^2} - (M_2g - M_2a) = -M_1a$$

Solve for a

$$I\frac{a}{r^2} + M_1a + M_2a = (M_2g)$$

$$a = \frac{M_2g}{(M_1 + M_2) + \frac{I}{r^2}}$$

Does the answer make sense?

$$a = \frac{M_2 g}{(M_1 + M_2) + \frac{I}{r^2}}$$

Check units

$$[a] = \frac{[kg][\frac{m}{s^2}]}{[kg] + [\frac{kg\ m^2}{m^2}]}$$

must add quantities with the same units

$$[a] = \frac{[kg][\frac{m}{s^2}]}{[kg] + [kg]}$$

$$[a] = [\frac{m}{s^2}]$$ **the units are fine**

Reasonable?

$$a = \frac{M_2 g}{(M_1 + M_2) + \frac{I}{r^2}}$$

yes, the block is not in free fall so its acceleration should be less than g

As the pulley gets more massive (I increases) the acceleration of the system decreases

makes sense

Suppose I = 0 (massless pulley) $a = \frac{M_2 g}{(M_1 + M_2)}$

Could do a quick calculation to check.

If we take the system to be both blocks connected by a massless rope

Net force = Weight of hanging block

$W_2 = M_2 g$

Mass of system is mass of both blocks

$M_1 + M_2$

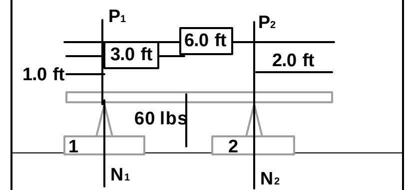
Newton's 2nd law

$M_2 g = (M_1 + M_2) a$

$$a = \frac{M_2 g}{(M_1 + M_2)}$$

Example

A uniform steel bar weighing 60 lbs is resting on two supports that are each sitting on a bathroom scale. The bar is 6.0 feet long. One support is 1.0 ft from the end of the bar and the other is 2.0 ft from the other end of the bar. What does each bathroom scale read?



Scale 1 reads P_1 and scale 2 reads P_2

Equal to forces on the bar N_1 and N_2 by Newton's 3rd law

System: bar

Use Newton's 2nd law to get N_1 and N_2

Forces

acceleration is zero

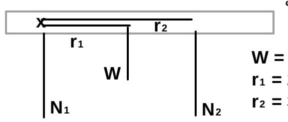
Torque

Choose axis at support 1

angular acceleration is zero

Free body diagram of bar

a **= 0**

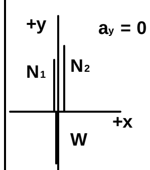


W = 60 lbs
$r_1 = 2.0$ **ft**
$r_2 = 3.0$ **ft**

Take axis at support 1 $S t = I a$

Take out as + $S t = - r_1 W + r_2 N_2 = 0$

Force diagram of bar



$a_y = 0$

Relate direction of a **to a**

angular acceleration + out

Consistent with

acceleration is + up

$S\ F_y = m a_y$

$S\ F_y = N_1 + N_2 - W = 0$

Target N_1, N_2

---

unknowns

Find N_1 | N_1

$S\ F_y = N_1 + N_2 - W = 0$

Find N_2

$S t = - r_1 W + r_2 N_2 = 0$ | N_2

2 unknowns, 2 equations

$- r_1 W + r_2 N_2 = 0$

$r_2 N_2 = r_1 W$

$$N_2 = \frac{r_1}{r_2} W$$

$$N_1 + \frac{r_1}{r_2} W - W = 0$$

$$N_1 = W(1 - \frac{r_1}{r_2})$$

$$N_1 = (60 lbs)(1 - \frac{2 ft}{3 ft})$$

$N_1 = 20$ **lbs**

$$N_2 = \frac{2 ft}{3 ft}(60\ lbs)$$

$N_2 = 40$ **lbs**

Could have taken axis of rotation at support 2 solve it that way

---

Direction of Torque

$t = r F_t$ $\quad å t = I a$

If there is only one torque

the direction of the torque is the direction of the angular acceleration

Get magnitude and direction from the force

Vector mathematics

Need to multiply 2 vectors

$\vec{r}$ $\quad \vec{F}$

to get a vector

$\vec{t}$

Vector cross product -- $\vec{t} = \vec{r} ´ \vec{F}$

$\vec{t} = \vec{r} ´ \vec{F}$ **means**

multiply r times the component of F perpendicular to r

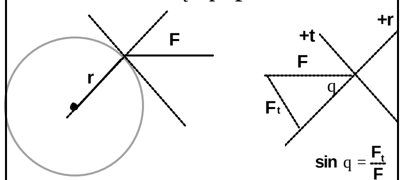
direction is perpendicular to both r and F

using the right hand rule
curl fingers from r to F

Thumb in direction of $\vec{t}$

Torque and the Cross Product

$\vec{\tau} = \vec{r} \times \vec{F}$



$\sin\theta = \frac{F_t}{F}$

Magnitude of torque

$\vec{r} \times \vec{F}$ means

multiply r times the component of F perpendicular to r

Tangential component of F is the component perpendicular to r

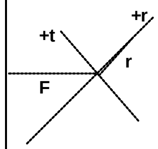
$\tau = r F_t$ Yes! What we wanted

If θ is the angle between radius from the axis of rotation and force

$\tau = r F \sin\theta$

Direction of torque

$\vec{\tau} = \vec{r} \times \vec{F}$



Right hand rule for vectors

Wrap fingers of right hand from first vector $\vec{r}$ to second vector $\vec{F}$

Thumb points in direction of result

out

Is this in the same direction as α ?

Yes

Notice that

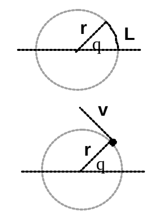
$\vec{r} \times \vec{F} = -(\vec{F} \times \vec{r})$

the order makes a difference in the vector cross product

Vectors and Rotational Kinematics

From the definition of angle

$\boxed{\theta = \frac{L}{r}}$



$\omega = \frac{d\theta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$

$\omega = \lim_{\Delta t \to 0} \frac{L}{r\Delta t}$

$\omega = \frac{1}{r} \lim_{\Delta t \to 0} \frac{L}{\Delta t}$

ω is out

$\omega = \frac{1}{r} v$

$\boxed{\omega r = v}$

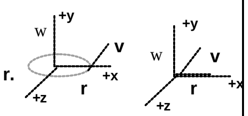
Direction of ω

Wrap fingers of right hand in direction of velocity

Thumb points in direction of ω

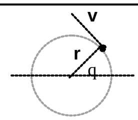
$\boxed{\vec{\omega} \times \vec{r} = \vec{v}}$

Wrap right hand fingers from ω to r. Thumb is in direction of v



$\alpha = \frac{d\omega}{dt}$

$\alpha = \frac{d\left(\frac{v}{r}\right)}{dt}$



$\alpha = \frac{1}{r}\frac{dv}{dt}$

If object speeding up

α is out

If object slowing down

α is in

$\alpha = \frac{1}{r} a_t$

$\boxed{\alpha r = a_t}$

Direction of α

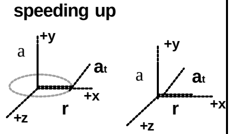
Wrap fingers of right hand in direction of tangential acceleration

Thumb points in direction of α

$\boxed{\vec{\alpha} \times \vec{r} = \vec{a}_t}$

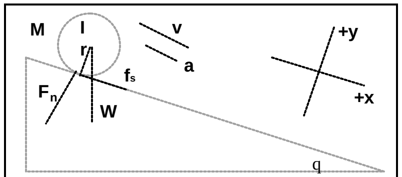
speeding up

Wrap right hand fingers from α to r. Thumb is in direction of a_t



Example

An object rolls down a ramp that is at a known angle with the horizontal. Determine its acceleration if you know its moment of inertia, its mass, and its radius.



What is the acceleration of the object?

What does rolling mean?

When the object goes around once, it has gone forward a distance of $2\pi r$

$r\Delta\phi = \Delta x$

which gives

$r\omega = v$ and $\alpha r = a$

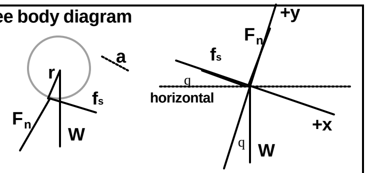
Need static friction for rolling

Use Newton's 2nd law

Forces

Torque

Free body diagram



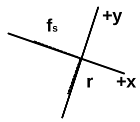
$$\sum F_x = W_x - f_s = Ma$$
$$Mg\sin\theta - f_s = Ma$$
$$\sum F_y = F_n - W_y = 0$$
$$F_n - Mg\cos\theta = 0$$

rolling

$$\alpha r = a$$

Torque

Choose axis of rotation as center of object



Only the frictional force causes a torque

$$\sum \tau = I\vec{\alpha}$$
$$\sum \vec{\tau} = r f_s \text{ (in)}$$

Target is a

---

		unknowns
Find a $Mg\sin\theta - f_s = Ma$	[1]	**a**
Find f_s $\tau = r f_s$	[2]	f_s
Find τ $\tau = I\alpha$	[3]	τ
Find α $\alpha r = a$	[4]	α

4 unknowns, 4 equations OK to solve

Solve [4] for α and put into [3]

$$\alpha r = a$$
$$\alpha = \frac{a}{r}$$
$$\tau = I\frac{a}{r} \quad \textbf{Into [2]}$$
$$I\frac{a}{r} = r f_s \quad \textbf{Solve for } f_s \textbf{ and put into [1]}$$
$$I\frac{a}{r^2} = f_s$$
$$Ma = Mg\sin\theta - I\frac{a}{r^2}$$
$$a\left(M + \frac{I}{r^2}\right) = Mg\sin\theta$$
$$\boxed{a = \frac{Mg\sin\theta}{M + \frac{I}{r^2}}}$$

How does this acceleration depend on the mass of the object?

$$\boxed{a = \frac{Mg\sin\theta}{M + \frac{I}{r^2}}}$$

since $I = f\,Mr^2$

where $f < 1$

$$a = \frac{Mg\sin\theta}{M + \frac{fMr^2}{r^2}}$$

$$\boxed{a = \frac{g\sin\theta}{1+f}}$$

The acceleration down the ramp does not depend on the object's mass.

The acceleration down the ramp also does not depend on the object's radius.

The acceleration down the ramp only depends on the object's shape (f).

Suppose that a hoop, a cylinder, and a solid sphere raced down a ramp. Which would win?