Motion of Complex Objects

Where are we?
In text (read, understand, do problems)

Read in text
Chapter 9
Chapter 8 (sec. 1-7)
Already did sec. 4 & 6
Read this week's lab

For the rest of the semester we will do the apply the physics we have learned to more realistic objects.

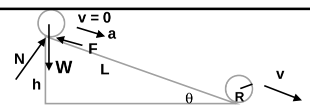
- **Describing the Motion of a System**
  - **Kinematics**
  - **Center of mass**
- **Conservation of Momentum for a system**
- **Conservation of Energy for a system**
- **Dynamics**

Rotations and Energy

This week
Solve a Problem using Energy
Be careful about Kinetic Energy
Describe rotational motion
Angles and speeds
Center of Mass
Shape is important
Moment of inertia
also called
Rotational inertia
Path
Chapter 9
Sections 5, 6, 1, 3
Chapter 8
Sections 1, 2, 5
Chapter 9
Sections 2, 4
Chapter 12
Sections 1 - 6

Example

You are the technical advisor on a new movie: "Indiana Jones Freezes in Minnesota." The script has a giant snowball rolling down a hill right at our hero. You know the length of the hill, its angle to the horizontal, and the radius of the snowball that starts from the top of the hill. The snowball rolls down the hill without slipping. Your job is to calculate the time it takes to get down the hill so that the stuntman knows when to dive out of the way.



Find time to get down hill for rolling.

Use kinematics to get the time from the change of velocity.

If all forces are constant during the motion, the acceleration is constant.
They are.

Use conservation of energy to get the change of velocity from top to bottom.

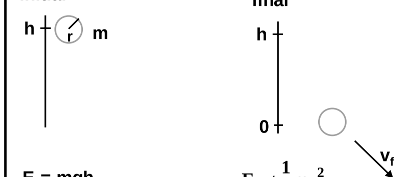
System: snowball and Earth
Initial time: snowball starts moving
Final time: snowball at bottom of hill

Snowball does not slide. It rolls.
F is the static frictional force

No change of internal energy (temperature)

Energy Diagram

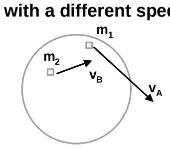
system: snowball, Earth



$E_i = mgh$ $\qquad E_f \neq \frac{1}{2}mv_f^2$

Evaluating the Kinetic Energy

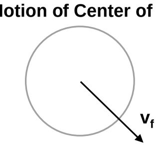
Parts of the snowball move with a different speeds



$$KE = \sum\left(\frac{1}{2}m_i v_i^2\right)$$

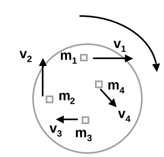
Easier to see if we separate center of mass motion and motion around center of mass

Motion of Center of Mass



$$KE_{com} = \frac{1}{2}Mv_f^2$$

If Center of Mass were not moving object would still have Kinetic Energy due to spinning



$$KE_{around\ com} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots$$

$$KE_{around\ com} = \sum\frac{1}{2}m_i v_i^2$$

$$KE = KE_{com} + KE_{around\ com}$$

Think Simple

Although every mass element of a rotating object has a different velocity

they are all moving around in the same way.

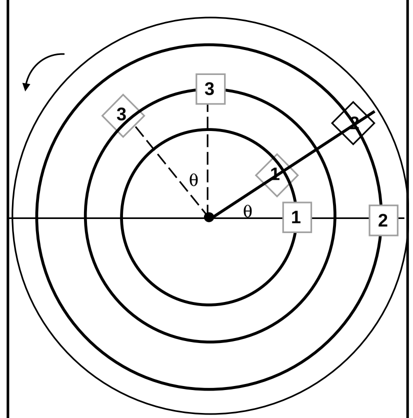
When one mass element rotates through a given angle, θ

all mass elements have rotated through the same angle, θ

Useful to describe the motion based on the behavior of the angle

Rotational Kinematics

Disk rotating around center



All mass elements go around by θ at the same time

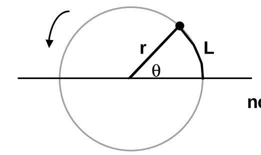
$v_2 > v_3 > v_1$

Radians

For a complete circle $L = 2\pi r$ $\frac{L}{r} = 2\pi$

For 1/2 a circle $L = \pi r$ $\frac{L}{r} = \pi$

For 1/10 a circle $L = \frac{2\pi r}{10}$ $\frac{L}{r} = \frac{2\pi}{10}$



$\theta = \frac{L}{r}$

no units for θ

θ is the fraction of circle for rotation: $\frac{\text{arc length}}{\text{radius}}$

$\theta_{\text{complete circle}} = 2\pi$ (1 revolution)

This gives the conversion between radians and degrees

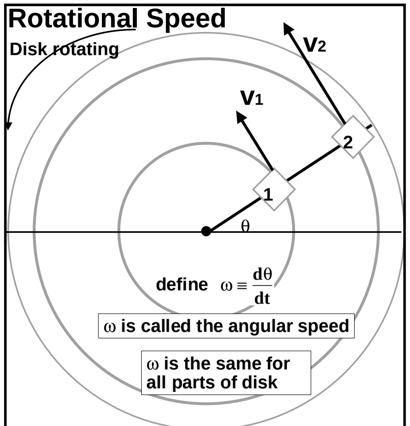
$\theta_{\text{complete circle}} = 360 \text{ degrees}$

2π = 360 degrees

$1 \text{ degree} = \frac{2\pi}{360} = \frac{\pi}{180}$ (radians)

Rotational Speed

Disk rotating



define $\omega \equiv \frac{d\theta}{dt}$

ω is called the angular speed

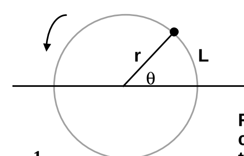
ω is the same for all parts of disk

Time for object 1 to go around by θ
equals
Time for object 2 to go around by θ

v_2 greater than v_1

ω_2 equal ω_1

Relationship of v to ω



$\theta = \frac{L}{r}$

Relate how θ changes with time to how L changes with time

$$\theta = \frac{1}{r}L$$

$$\frac{d\theta}{dt} = \frac{1}{r}\frac{dL}{dt}$$

$$\frac{dL}{dt} = v$$

$$\frac{d\theta}{dt} = \frac{1}{r}v$$

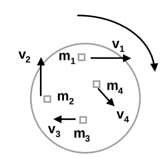
$$\frac{d\theta}{dt} = \omega$$

Units of ω are $\frac{1}{\text{sec}}$

$$\omega = \frac{v}{r} \qquad \frac{\left[\frac{m}{s}\right]}{[m]} = \frac{1}{[s]}$$

Rotational Kinetic Energy

If Center of Mass were not moving object would still have Kinetic Energy due to spinning



$$KE_{\text{around com}} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots$$

$$KE_{\text{around com}} = \sum \frac{1}{2}m_iv_i^2$$

$$KE_{\text{around com}} = \sum \frac{1}{2}m_ir_i^2\omega^2$$

$$KE_{\text{around com}} = \frac{1}{2}\left(\sum m_ir_i^2\right)\omega^2$$

$$KE_{around\ com} = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

$$\sum m_i r_i^2$$

Describes on how object"s mass is arranged

Its shape

A property of the object

Does not depend on the object's motion

Give it a name --- Moment of Inertia

$$I \equiv \sum m_i r_i^2$$

units [kg][m²]

For a continuous object

Make mass elements very small and all the same

$$m_i \xrightarrow{m_i \to 0} dm$$

$$I \equiv \int_{over\ object} r^2 dm$$

Rotational Kinetic Energy

$$KE_{around\ com} = \frac{1}{2} I\omega^2$$

Total Kinetic Energy

$KE = KE_{com} + KE_{around\ com}$

$$KE_{com} = \frac{1}{2} M v_f^2$$

$$KE_f = \frac{1}{2} M v_f^2 + \frac{1}{2} I\omega_f^2$$

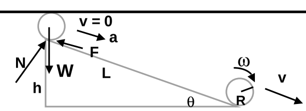
Now we can solve the original problem if we know

The shape of the object or its moment of inertia

The relationship between ω and v_{com} for a rolling object

he Problem with More Information

You are the technical advisor on a new movie: "Indiana Jones Freezes in Minnesota." The script has a giant snow ball rolling down a hill right at our hero. You know the length of the hill, its angle to the horizontal, and the radius of the snowball that starts from the top of the hill. The snowball rolls down the hill without slipping. Your job is to calculate the time it takes to get down the hill so that the stuntman knows when to dive out of the way. Since the snowball is solid,its moment of inertia about its center of mass is a fraction f of a ring about its center. You will determine that fraction later.



Find time to get down hill for rolling.

Since all forces are constant during the motion we have constant acceleration.

Use conservation of energy to get the change of velocity from top to bottom.

System: snowball and Earth

Initial time: snowball starts moving

Final time: snowball at bottom of hill

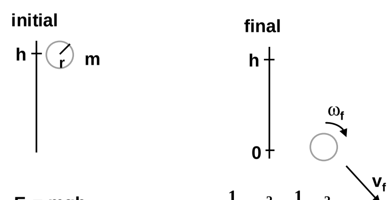
Use constant acceleration kinematics to get the time.

Assume snowball does not slide. It rolls.

F is the static frictional force

No change of internal energy (temperature)

Energy Diagram for rolling



$E_i = mgh$

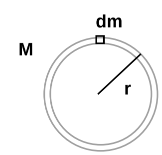
$$E_f = \frac{1}{2} m v_f^2 + \frac{1}{2} I\omega_f^2$$

$E_{transfer} = 0$

$I = fI_{ring}$

What is the moment of inertia of a ring about its center?

Moment of Inertia of a Ring about its center



$$I = \int_{over\ object} r^2 dm$$

r is the same for all dm in a ring

$$I = r^2 \int_{over\ object} dm$$

$$M = \int_{over\ object} dm$$

$I = Mr^2$ for a ring rotating about its center

$I_{snowball} = f\,Mr^2$

Check tables in text for different shaped objects

How to calculate f -- Later

Rolling

What is the relationship between

ω_f and v_f ?

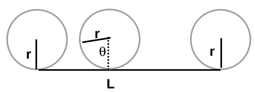
Usually NONE!!

But

There is when an object rolls without slipping

For rolling, the angle of rotation is related to the distance traveled

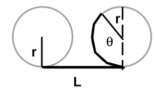
When rolling, if the object make one complete revolution, how far does it go?



L = Circumference = $2\pi r$

Object rolls through an angle θ

How far does it go?



L = Arc length = $r\theta$

How fast the object roll?

$L = r\theta$

How does the distance rolled change with time

$$\frac{dL}{dt} = r\frac{d\theta}{dt}$$

What do the terms mean?

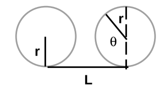
$\frac{dL}{dt}$ How fast the object goes forward

$$\frac{dL}{dt} = v_{com}$$

$\frac{d\theta}{dt}$ How fast the object goes around

$$\frac{d\theta}{dt} = \omega$$

Review



$L = r\theta$

$$\frac{dL}{dt} = r\frac{d\theta}{dt}$$

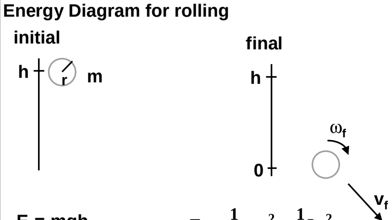
$$\frac{dL}{dt} = v_{com}$$

$$\frac{d\theta}{dt} = \omega$$

$v_{com} = r\omega$ for rolling without slipping

Back to the Problem

Energy Diagram for rolling



$E_i = mgh$ $E_f = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2$

$E_{transfer} = 0$

$I = fI_{ring}$ $\omega_f r = v_f$ for rolling

Conservation of energy

$$\Delta E_{system} = \Delta E_{transfer}$$

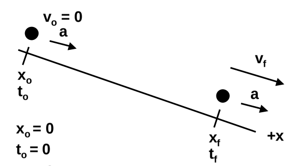
$$\frac{1}{2}mv_f^2 + \frac{1}{2}\left(fmR^2\right)\left(\frac{v_f}{R}\right)^2 - mgh = 0$$

$$v_f^2 + fv_f^2 - 2gh = 0$$

Need to get the time from kinematics

Constant Acceleration Kinematics

Motion Diagram:



$x_o = 0$
$t_o = 0$
$v_o = 0$
a = ?

$x_f = L$
$t_f = ?$
$v_f = ?$

For constant acceleration

$$x_f = \frac{1}{2}a(t_f - t_o)^2 + v_o(t_f - t_o) + x_o = \frac{1}{2}at_f^2$$

$$a = a_{ave} = \frac{v_f - v_o}{t_f - t_o} = \frac{v_f}{t_f}$$

Target: t_f

---

		unknowns
Find t_f		t_f
$L = \frac{1}{2}at_f^2$	1	a
Find a		
$a = \frac{v_f}{t_f}$	2	v_f
Find v_f		
$v_f^2 + fv_f^2 - 2gh = 0$	3	h
Find h		
$\sin\theta = \frac{h}{L}$	4	

4 equations for 4 unknowns ok

4 $L \sin\theta = h$ Put into [3] to solve for v_f

$$v_f^2 + f v_f^2 - 2gL \sin\theta = 0$$

$$(1+f)v_f^2 = 2gL \sin\theta = 0$$

$$v_f = \sqrt{\frac{2gL \sin\theta}{(1+f)}} = 0$$ Put into [2] to solve for a

$$a = \frac{\sqrt{\frac{2gL \sin\theta}{(1+f)}}}{t_f}$$ Put into [1] to solve for t_f

$$L = \frac{1}{2}\frac{\sqrt{\frac{2gL \sin\theta}{(1+f)}}}{t_f} t_f^2$$

$$\boxed{L\sqrt{\frac{2(1+f)}{gL \sin\theta}} = t_f}$$

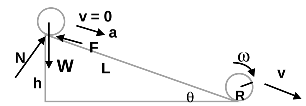
check units:

$$\sqrt{\frac{[m]}{\left[\frac{m}{s^2}\right]}} = [s]$$ ok

$$\boxed{\sqrt{\frac{2L(1+f)}{g \sin\theta}} = t_f}$$

Another Question - Same Problem

What is the snowball's acceleration?



Use: $L = \frac{1}{2} a t_f^2$

$$L = \frac{1}{2} a \left(\sqrt{\frac{2L(1+f)}{g \sin\theta}}\right)^2$$

$$L = \frac{1}{2} aL \frac{2(1+f)}{g \sin\theta}$$

$$\frac{g \sin\theta}{(1+f)} = a$$ Why is this reasonable?

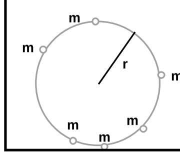
Moment of Inertia (review)

The moment of inertia of an object depends only on its geometry and the location of the axis of rotation (pivot point)

Add up mass elements (integrate) but the influence of each mass element depends on its distance from the axis of rotation.

Example, find the moment of inertia:

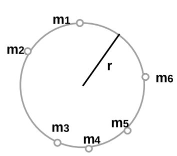
6 objects going around a point at equal distance from that point.



$$\left(\sum_i m_i r_i^2\right) \equiv I$$

I = ?

6 objects of total mass M going around a point at equal distance from that point.



$$\left(\sum_i m_i r_i^2\right) \equiv I$$

$$\left(\sum_i m_i r^2\right) = I$$ All masses are the same distance from the axis of rotation

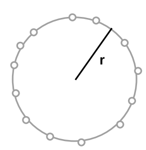
$$\left(\sum_i m_i\right) r^2 = I$$ Factor out the common r

$I = (m_1 + m_2 + m_3 + m_4 + m_5 + m_6)\, r^2$

If all of the objects have equal mass

$I = 6\, m\, r^2$

A large number of objects of total mass M going around a point at equal distance from that point.



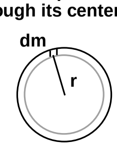
$$\left(\sum_i m_i r_i^2\right) \equiv I$$

$$\left(\sum_i m_i r^2\right) = I$$

$$\left(\sum_i m_i\right) r^2 = I$$

$$Mr^2 = I$$

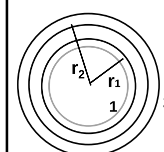
A thin hoop of mass M rotating about an axis through its center



$$\left(\sum_i dm_i\right) r^2 = I$$

$$Mr^2 = I$$

Two thin hoops of total mass M rotating about an axis through their center



$I = m_2 r^2 + m_1 r_1^2$

$M = m_2 + m_1$

2 If all of the mass were at the outer hoop

$$Mr^2 = I$$

Since some of the mass is at a smaller radius

$$I < Mr^2$$

In general

$I = f\, Mr^2$

where f is a fraction less than 1

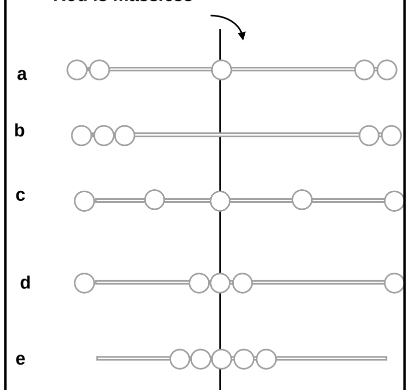
r is the distance from the axis of rotation to the farthest point on the object from that axis.

See table on page 249 of text

Order the objects from largest to smallest moment of inertia

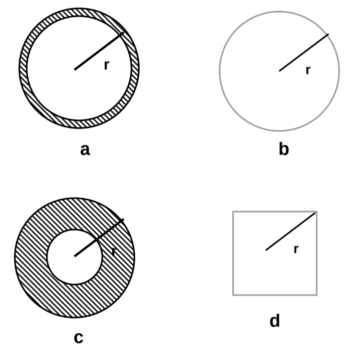
Each ball has mass M

Rod is massless



Order the objects from largest to smallest moment of inertia

All objects have mass M



Review

- Angular kinematics
  - $\theta = L / r$
  - $\omega = v / r$
  - $\alpha = a / r$
- Moment of Inertia
  - $I = \Sigma m_i r_i^2$
    - $I = M r^2$ (ring)
    - $I = f M r^2$
- Rolling without slipping
  - $v_{com} = \omega r$
- Rotational Kinetic Energy
  - $KE_{rotation} = (1/2) I \omega^2$