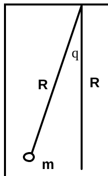
Example:

When visiting an office building you notice that it has an atrium with an enormous pendulum with one end attached to the roof of the building and the other end nearly reaching the floor. You observe that the pendulum takes 6.0 seconds to swing across the floor. How tall is the building?



What is the length of the pendulum?

Period = T = 12.0 s

Use dynamics to get equation of motion of pendulum bob.

Use torques : axis of rotation is roof suspension

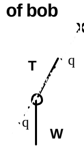
$$\sum \vec{\tau} = I\vec{\alpha}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

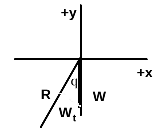
Assume:

- No torques due to friction in suspension
- No air resistance
- Bob is a point object, massless string

Free body diagram of bob



Torque diagram



I = mR²

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = -Rmg\sin\theta = I\frac{d^2\theta}{dt^2}$$

Target R:

Find R

$$-Rmg\sin\theta = mR^2\frac{d^2\theta}{dt^2}$$

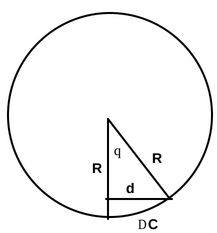
$$-g\sin\theta = R\frac{d^2\theta}{dt^2}$$

Difficult to guess solution to this equ.

Make an approximation

If angle is small

$\sin\theta = \theta$ **(in radians)**



$$\theta = \frac{\Delta C}{R}$$

$$\sin\theta = \frac{d}{R}$$

for small θ

$\Delta C \rightarrow d$

so

$\sin\theta \rightarrow \theta$

$$-g\sin\theta = R\frac{d^2\theta}{dt^2}$$ **for small angles becomes**

$$-g\theta = R\frac{d^2\theta}{dt^2}$$

guess periodic solution as

$$\theta = A\cos(2\pi ft + \phi)$$

$$\frac{d\theta}{dt} = -A\sin(2\pi ft + \phi)2\pi f$$

$$\frac{d^2\theta}{dt^2} = -A\cos(2\pi ft + \phi)(2\pi f)^2$$

$$-gA\cos(2\pi ft + \phi) = -RA\cos(2\pi ft + \phi)(2\pi f)^2$$

$$g = R(2\pi f)^2$$

f = 1/T

$$g = R\left(2\pi\frac{1}{T}\right)^2$$

$$\boxed{g\left(\frac{T}{2\pi}\right)^2 = R}$$

$$g\left(\frac{T}{2\pi}\right)^2 = R$$

check units:

$$\left[\frac{m}{s^2}\right][s]^2 = [m] \qquad \textbf{ok}$$

$$32\frac{ft}{s^2}\left(\frac{12.0s}{2\pi}\right)^2 = R$$

$$\boxed{117\text{ ft} = R}$$

Is this unreasonable?

A longer length takes more time which is reasonable.

At 10 ft per story, this is a 12 story building

Tallest buildings in Minneapolis are about 50 floors high.

This is not unreasonable.

Note that frequency of pendulum

from torque equation

$$g = R(2\pi f)^2$$

$$\frac{1}{2\pi}\sqrt{\frac{g}{R}} = f$$

frequency of oscillation of pendulum

Independent of mass

Depends on g

If you recall

frequency of oscillation of spring

$$\frac{1}{2\pi}\sqrt{\frac{k}{m}} = f$$

Independent of g

Depends on mass

Choose the approach to the problem

Several are possible

Conservation Approach

- energy
- momentum
- angular momentum

Dynamics Approach

- forces
- torques

How to choose

Do you have enough information to use the approach?

Does the approach give you what you want to know?

Do the same problem but now use a conservation of energy approach.

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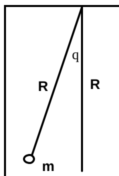


What is the length of the pendulum?

Period = T = 12.0 s

Use conservation of energy to get equation of motion of pendulum bob.

System: bob & Earth

Initial time: bob at highest point

Final time: during swing

Types of energy to consider

Gravitational potential energy

Rotational kinetic energy

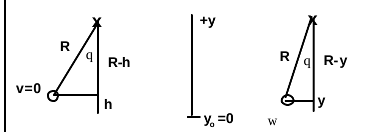
Assume:

No energy transfer due to friction in suspension or air resistance

Massless string

System: bob & Earth

Initial Energy   Final Energy



$E_i = mgh$ $E_{transfer} = 0$ $E_f = \frac{1}{2}I\omega^2 + mgy$

Conservation of Energy:

$$E_f - E_i = E_{transfer}$$

$$\frac{1}{2}I\omega^2 + mgy - mgh = 0$$

$\omega = \frac{d\theta}{dt}$ $I = mR^2$

need y in terms of θ

$\cos\theta = \frac{R - y}{R}$ $y = R - R\cos\theta$

Conservation of energy:

$$\frac{1}{2}I\omega^2 + mgy - mgh = 0$$

$$\frac{1}{2}mR^2\left(\frac{d\theta}{dt}\right)^2 + mg(R - R\cos\theta) - mgh = 0$$

$$\frac{1}{2}R^2\left(\frac{d\theta}{dt}\right)^2 - gR\cos\theta + gR - gh = 0$$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$ For small angles

$$\frac{1}{2}R^2\left(\frac{d\theta}{dt}\right)^2 - gR\left(1 - \frac{\theta^2}{2}\right) + gR - gh = 0$$

$$\frac{1}{2}R^2\left(\frac{d\theta}{dt}\right)^2 + gR\frac{\theta^2}{2} - gh = 0$$

$$R^2\left(\frac{d\theta}{dt}\right)^2 + gR\theta^2 - 2gh = 0$$

For a periodic solution

Guess

$\theta = A\sin(2\pi f t + \phi)$

Plug it in to check

$$R^2\left(\frac{d\theta}{dt}\right)^2 + gR\theta^2 - 2gh = 0$$

$$\frac{d\theta}{dt} = A\cos(2\pi f t + \phi)b$$

$$R^2(Ab\cos(2\pi f t + \phi))^2 + gR(A\sin(2\pi f t + \phi))^2 - 2gh = 0$$

$$A^2(2\pi f)^2R^2\cos^2(2\pi f t + \phi) + A^2 gR\sin^2(2\pi f t + \phi) - 2gh = 0$$

$$A^2(2\pi f)^2R^2\cos^2(2\pi f t + \phi) + A^2 gR\sin^2(2\pi f t + \phi) = 2gh$$

Only possible if

$$A^2(2\pi f)^2R^2\cos^2(2\pi f t + \phi) + A^2 gR\sin^2(2\pi f t + \phi)$$

Does not depend on time

$$\cos^2(2\pi f t + \phi) + \sin^2(2\pi f t + \phi) = 1$$

This is independent of t

Will result from

$$A^2(2\pi f)^2R^2\cos^2(2\pi f t + \phi) + A^2 gR\sin^2(2\pi f t + \phi)$$

if

$$A^2(2\pi f)^2R^2 = A^2 gR$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{R}}$$

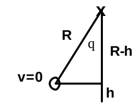
$$A^2\frac{g}{R}R^2\cos^2(2\pi f t + \phi) + A^2 gR\sin^2(2\pi f t + \phi) = 2gh$$

$$A^2 gR = 2gh$$

$$A = \sqrt{\frac{2h}{R}}$$

Check

$A = \sqrt{\frac{2h}{R}}$ **A is θ_{max}**



For small angles, $\sin\theta = \theta$

$$\theta_{max} = \frac{\sqrt{R^2 - (R-h)^2}}{R} = \frac{\sqrt{2Rh - h^2}}{R} = \frac{\sqrt{h(2R-h)}}{R}$$

For small θ, 2R>>h

$$\frac{\sqrt{h(2R-h)}}{R} = \frac{\sqrt{h(2R)}}{R} = \sqrt{\frac{2h}{R}}$$

This agrees with the solution

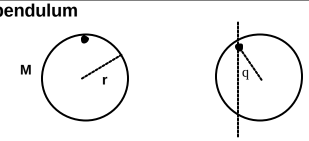
$$A = \sqrt{\frac{2h}{R}}$$

$\theta = A\sin(2\pi f t + \phi)$ $\quad f = \frac{1}{2\pi}\sqrt{\frac{g}{R}}$

Example

Your friend is designing a decorative clock for a class project. The plan calls for the timing of the clock to be regulated by a swinging ring. The ring is suspended at its top so that it swings back and forth. Your friend asks you to determine how the size and mass of the ring affects the timing of the clock.

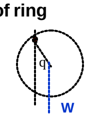
A pendulum



Find the frequency of a ring of mass M and radius R.

Use torques

Free body diagram of ring



$$\sum \tau = I\alpha$$

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = -rMg\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$-rMg\theta = I\frac{d^2\theta}{dt^2}$$

$-rMg\theta = I\frac{d^2\theta}{dt^2}$ **for small angles**

guess periodic solution as

$\theta = A\cos(2\pi f t + \phi)$

$$\frac{d\theta}{dt} = -A\sin(2\pi f t + \phi)2\pi f$$

$$\frac{d^2\theta}{dt^2} = -A\cos(2\pi f t + \phi)(2\pi f)^2$$

$$-rMgA\cos(2\pi f t + \phi) = -IA\cos(2\pi f t + \phi)(2\pi f)^2$$

$$rMg = I(2\pi f)^2$$

f = 1/T

$$\frac{1}{2\pi}\sqrt{\frac{rMg}{I}} = f$$

Find I

Find I for ring about center of mass

Use that to find I about axis of rotation

Parallel axis theorem

$$I = Md^2 + I_{com}$$

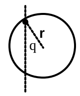
Moment of inertia around the center of mass

$$I_{com} = Mr^2$$

plus moment inertia of center of mass

$$d = r$$

For ring pendulum



$$\frac{1}{2\pi}\sqrt{\frac{rMg}{I}} = f$$

$$I = Mr^2 + Mr^2 = 2Mr^2$$

$$\frac{1}{2\pi}\sqrt{\frac{rMg}{2Mr^2}} = f$$

$$\frac{1}{2\pi}\sqrt{\frac{g}{2r}} = f \qquad 2\pi\sqrt{\frac{2r}{g}} = T$$

Compare string pendulum with ring pendulum

For what length of string pendulum do you get the same period as a ring pendulum?

String

$$-g\sin\theta = R\frac{d^2\theta}{dt^2}$$

$$\frac{1}{2\pi}\sqrt{\frac{g}{R}} = f$$

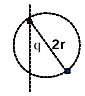
$$2\pi\sqrt{\frac{R}{g}} = T$$

Ring

$$-rMg\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{1}{2\pi}\sqrt{\frac{g}{2r}} = f$$

$$2\pi\sqrt{\frac{2r}{g}} = T$$

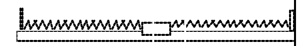


Same period if
$R = 2r$

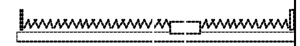
As in lab determine the motion of the object

Glider on airtrack connected to two springs as an example

At equilibrium



Displace from equilibrium and release



Know:

mass of glider : m_g

spring constant of each spring : k_1, k_2

Question:

What is position of glider as a function of time?

Use:

conservation of energy:

System: glider & springs

Initial time: just after release

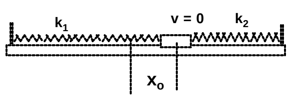
Final time: any time after that

Assume:

no friction

massless springs

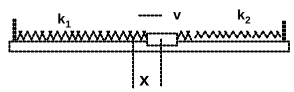
Initial time



E_i = PE (spring 1) + PE (spring 2)

$$E_i = \frac{1}{2}k_1x_o^2 + \frac{1}{2}k_2x_o^2$$

Final time



E_f = PE (spring 1) + PE (spring 2) + KE

$$E_f = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2 + \frac{1}{2}mv^2$$

Conservation of Energy

$$E_f - E_i = E_{transfer} = 0$$

$$\frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2 + \frac{1}{2}mv^2 - \left(\frac{1}{2}k_1x_o^2 + \frac{1}{2}k_2x_o^2\right) = 0$$

$$(k_1 + k_2)x^2 + mv^2 - (k_1 + k_2)x_o^2 = 0$$

$$(k_1 + k_2)x^2 + m\left(\frac{dx}{dt}\right)^2 - (k_1 + k_2)x_o^2 = 0$$

Periodic solution - not obvious

$$(k_1 + k_2)x^2 + m\left(\frac{dx}{dt}\right)^2 - (k_1 + k_2)x_o^2 = 0$$

$$x = A\cos(bt + \phi)$$

$\cos(bt+\phi)$ repeats in one period

$$\cos(bt+\phi) = \cos(b(t+T)+\phi)$$

$$bt + \phi + 2\pi = bt + bT + \phi$$

$$2\pi = bT$$

$$b = \frac{2\pi}{T}$$

$$b = 2\pi f$$

$$\frac{dx}{dt} = -A\sin(bt+\phi)b$$

$$(k_1 + k_2)(A\cos(bt+\phi))^2 + m(-A\sin(bt+\phi)b)^2 - (k_1 + k_2)x_o^2 = 0$$

$$(k_1 + k_2)A^2 \cos^2(bt + f) + mA^2b^2 \sin^2(bt + f) - (k_1 + k_2)x_o^2 = 0$$

$$(k_1 + k_2)A^2 \cos^2(bt + f) + mA^2b^2\left(1 - \cos^2(bt + f)\right) - (k_1 + k_2)x_o^2 = 0$$

$$(k_1 + k_2)A^2 \cos^2(bt + f) - mA^2b^2 \cos^2(bt + f) + mA^2b^2 - (k_1 + k_2)x_o^2 = 0$$

Part of equation varies with time

that part = 0

Part of equation constant

that part = 0

Time dependent part

$$(k_1 + k_2)A^2 - mA^2b^2 = 0$$

$$(k_1 + k_2) = mb^2$$

$$\sqrt{\frac{(k_1 + k_2)}{m}} = b$$

$$\boxed{\frac{1}{2p}\sqrt{\frac{(k_1 + k_2)}{m}} = f}$$

same as with force approach !!

Time independent part of equation

$$mA^2b^2 = (k_1 + k_2)x_o^2$$

$$mA^2\frac{(k_1 + k_2)}{m} = (k_1 + k_2)x_o^2$$

$A = x_o$ **You knew that**

solution to energy equation:

$$x = x_o \cos(2pft + f)$$

What about f**?**

At t=0, $x = x_o$

$$x_o = x_o \cos(0 + f)$$

$$f = 0$$

$$x = x_o \cos(2pft)$$