Picking Coordinate Axes

If the object you are interested in

Is accelerating

Choose one axis along the acceleration

Sum of Force components along that axis

equals ma

Sum of Force components along any other axis

equals 0

Calculations are easier

no components of acceleration to deal with

Example:

Car going around a banked corner.
Corner has a known angle and radius

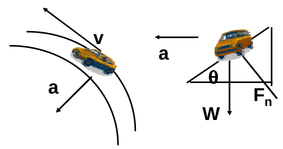
No friction, horizontal path

What is car's speed?

Motion

Top view

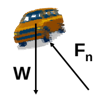
Front view



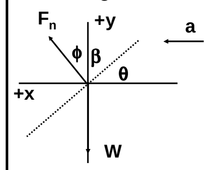
Radial acceleration gives car's speed.

Get acceleration from forces

Free body Diagram of car



Force Diagram of car



$$\sum F_x = F_{nx}$$

$$\sum F_y = F_{ny} - W$$

Horizontal Direction

$\sum F_x = ma_x$

$F_{nx} = ma_x$

$a_x = \frac{v^2}{r}$

Vertical Direction

$\sum F_y = ma_y$

$F_{ny} - W = ma_y$

$a_y = 0$

$W = mg$

$F_{ny} - mg = 0$

Components of F_n

$\frac{F_{nx}}{F} = \sin\phi$ $\quad \frac{F_{ny}}{F} = \cos\phi$

$\phi + \beta = 90^o$ $\quad \theta + \beta = 90^o$

$\theta = \phi$

Target quantity is v

PLAN		unknowns
Find v		v
For circular motion get speed from radial acceleration		
$a = \frac{v^2}{r}$	1	a
Find a		
$F_{nx} = ma$	2	m, F_{nx}
Find F_{nx}		
$F_{nx} = F_n \sin\theta$	3	F_n
Find F_n		
$F_{ny} = F_n \cos\theta$	4	F_{ny}
Find F_{ny}		
$F_{ny} - mg = 0$	5	

5 equations, 6 unknowns

$$F_{ny} = mg$$

$$mg = F_n \cos\theta$$

$$\frac{mg}{\cos\theta} = F_n$$

$$F_{nx} = \frac{mg}{\cos\theta}\sin\theta$$

$$\frac{mg}{\cos\theta}\sin\theta = ma$$

$$\frac{g}{\cos\theta}\sin\theta = a$$

$$\frac{g}{\cos\theta}\sin\theta = \frac{v^2}{r}$$

$$\boxed{\sqrt{gr\frac{\sin\theta}{\cos\theta}} = v}$$

check units

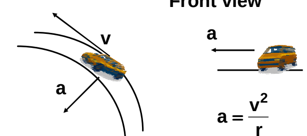
If road were not banked, could a car go around a curve?

No friction, horizontal path

Motion

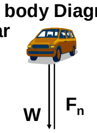
Top view

Front view



Forces (no friction case)

Free body Diagram of car



No force in desired direction

radial inward

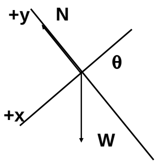
Therefore no acceleration

radial inward

Car cannot go around the curve !!

Using a different coordinate system

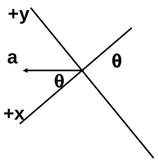
Force Diagram of car



$$\sum F_x = W_x = ma_x$$

$$\sum F_y = N - W_y = ma_y$$

$$a = \frac{v^2}{r}$$



$$\cos\theta = \frac{a_x}{a}$$

$$\sin\theta = \frac{a_y}{a}$$

$$a_x^2 + a_y^2 = a^2$$

Target quantity is v

PLAN

	unknowns
Find v	v
$a = \frac{v^2}{r}$	a
Find a	
$\cos\theta = \frac{a_x}{a}$	a_x
Find a_x	
$N_x = ma_x$	m, N_x
Find N_x	
$N_x = N\cos\theta$	
Find N	N
$N_y = N\sin\theta$	N_y
Find N_y	
$ma_y = N_y - mg$	a_y
Find a_y	
$\sin\theta = \frac{a_y}{a}$	

7 equations, 8 unknowns

Now add friction

Don't want car to slide

static friction

Which direction?

For static friction

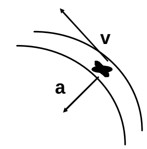
you CANNOT use direction of motion to give direction of frictional force

Case I: Flat road

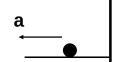
horizontal path

Motion

Top view

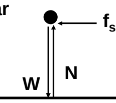


Front view



$$a = \frac{v^2}{r}$$

Free body Diagram of car



For car to go around curve

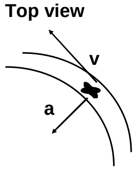
Static frictional force must be radial

Case II: Banked curve

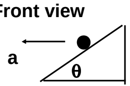
Car going around a banked corner horizontal path

Motion

Top view



Front view



$$a = \frac{v^2}{r}$$

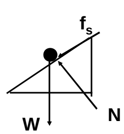
Forces

To help, the static frictional force must have a radial component

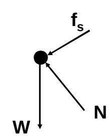
At high speeds the car will slide off the top of the banked curve

If this is not to happen

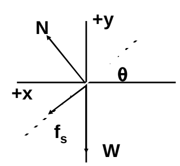
Static frictional force is down the bank



Free body Diagram of car



Force Diagram of car

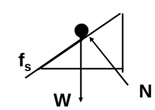


$$\sum F_x = N_x + f_{sx}$$

$$\sum F_y = N_y - W - f_{sy}$$

This will give the car's maximum speed to stay in a horizontal path (no skidding)

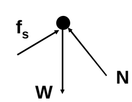
At low speeds the car will slide down the banked curve



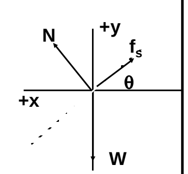
If this is not to happen

Static frictional force is up the bank

Free body Diagram of car



Force Diagram of car



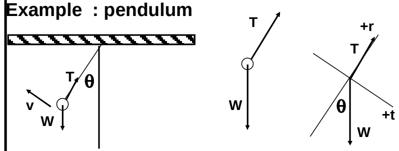
$$\sum F_x = N_x - f_{sx}$$

$$\sum F_y = N_y - W + f_{sy}$$

This will give the car's minimum speed to stay in a horizontal path (no skidding

Non constant Forces

Example : pendulum



Motion in t direction: $\Sigma F_t = ma_t$

$W_t = ma_t$, a_t ° 0

$W \sin\theta = m\, a_t$

$mg \sin\theta = m\, a_t$

$g \sin\theta = a_t$

The gravitational force, W, causes the tangential acceleration.

The tangential acceleration changes because the component of W in the tangential direction changes

$\Sigma F_t = W \sin\theta$

Motion in r direction: $\Sigma F_r = ma_r$

$T - W_r = ma_r$ a_r ° 0

$T - W\cos\theta = ma_r$

$\frac{T}{m} - g\cos\theta = a_r$

From definition of acceleration and velocity for circular motion

$a_r = \frac{v^2}{r}$

The combination of the gravitational force, W, and the force of the string on the ball, T, cause the radial acceleration.

The radial acceleration changes because the component of W in the radial direction changes, and T changes

$\Sigma F_r = T - W\cos\theta$

$T = m(g\cos\theta + \frac{v^2}{r})$

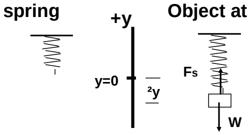
FORCES WHICH CHANGE AS OBJECT MOVES

Object on a Spring

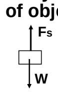
No motion (especially no acceleration)

Unstretched spring

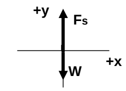
Object at rest



Free-body diagram of object



Force diagram of object



since $a_y = 0$, $\Sigma F_y = 0$

$\Sigma F_y = F_s - W$

$\Sigma F_y = ma_y = 0$

$F_s = W$

This tells us the value of F_s

IN THIS CASE ONLY

How does the spring force behave:

If you increase the weight of the object

Δy increases

Double the weight

doubles Δy

Theory of the spring force

$F_s = k\, y$ y measured from unstretched position

As the object gets further away from the unstretched position, the force increases.

Position change could be Δy or Δx

Call it Δx

$F_s = kx$

if x is measured from the unstretched position

Direction of that force is always opposite to the direction of the displacement from the unstretched position

The force that a spring exerts on an object increases with its displacement from its unstretched position

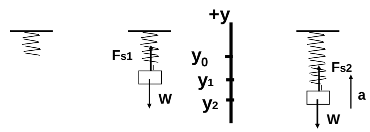
---

Object in motion on a vertical Spring

Assume spring is massless

Determine the object's acceleration

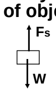
spring | Object at rest | Pull object down and let go



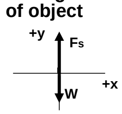
Use relationship between force and accel.

Newton's 2nd law

Free-body diagram of object



Force diagram of object



Target quantity: a_y

$\Sigma F_y = ma_y$ $F_s = k\,\Delta y$

Plan

Find a_y

$\Sigma F_y = ma_y$

Find ΣF_y

$\Sigma F_y = F_{s2} - W$

Find F_{s2}

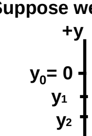
$F_{s2} = k\,(y_2 - y_0)$

$\Sigma F_y = k\,(y_2 - y_0) - W$

$k(y_2 - y_0) - W = ma_y$

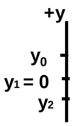
$\frac{k\,(y_2 - y_0) - W}{m} = a_y$

Suppose we choose the origin at y_0



$\frac{k\, y_2 - W}{m} = a_y$

Suppose we choose the origin at y_1



$$\frac{k\,(y_2 - y_0) - W}{m} = a_y$$

$$\frac{k\,(y_2 + (y_1 - y_1) - y_0) - W}{m} = a_y$$

$$\frac{k\,(y_2 - y_1) + k\,(y_1 - y_0) - W}{m} = a_y$$

What is $k\,(y_1 - y_0)$?

If the object hangs at rest on the spring

$F_{s1} = k\,(y_i - y_0)$

$\Sigma F_y = F_{s1} - W$

since $a_y = 0$, $\Sigma F_y = 0$

$F_{s1} = W$

$W = k\,(y_i - y_0)$

$$\frac{k\,(y_2 - y_1) + W - W}{m} = a_y$$

$$\frac{k\,(y_2 - y_1)}{m} = a_y$$

The acceleration of the object does not depend on its weight

Surprises from the fundamental concepts of Force and acceleration

For an object hanging vertically on a spring

If you set it into motion

$$a_y = \frac{k}{m}\,y'$$

y' is displacement from equilibrium position of the object

The acceleration of the object does not depend on g

The gravitational force does not affect the object's motion

Except, of course, for displacing the equilibrium point.

Position from which displacement is measured

Spring

Useful to measure the force on an object

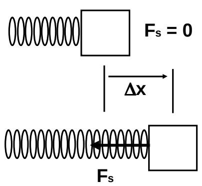
- Hang an object on a spring
  - Equilibrium position gives object's weight

Useful to measure the mass of an object

- Let spring oscillate
  - Acceleration gives object's mass

Useful to measure the acceleration of a system

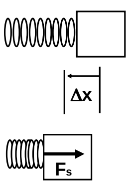
Stretched spring exerts a force on an object



$F_s = 0$ (equilibrium position)

Whether force is + or - depends on your coordinate system

Compressed spring exerts a force on an object



$F_s = 0$ (equilibrium position)

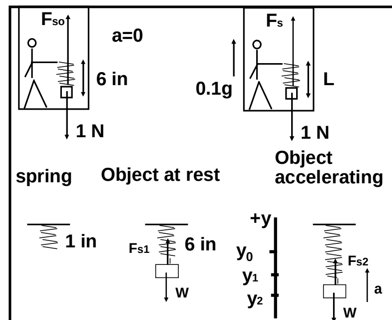
Whether force is + or - depends on your coordinate system

Force is always in opposite direction from the displacement from equilibrium position

---

Example

Your assignment is to design a simple, hand held device to measure acceleration. Your design is a spring which you hold on one end so that it hangs vertically with a 1.0 N object on the other end. To test the device you take it to the elevators in the IDS building where, you have been told, the elevators' maximum acceleration is 0.10g. Before the elevator starts, you hang the object on the spring and it stretches from 1 inch to 6 inches. What is the length of the spring for the elevator's maximum acceleration?



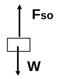
What is length of spring for accelerating object ?

Use relationship between force and accel.

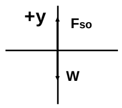
Newton's 2nd law

Use force law for spring

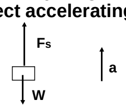
Free body diagram object at rest



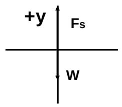
Force diagram object at rest



Free body diagram object accelerating



Force diagram object accelerating



W = 1.0 N

a = 0.1 g

$y_o = 0$

$y_1 = -6$ in

Find y_2

Use

$\sum F_y = ma_y$ $\quad F_s = k\Delta y$

Plan — **unknowns**

Find y_2 — y_2

accelerating object

$F_s = -k(y_2 - y_o)$ — F_s, k

$F_s = -ky_2$ [1]

Find F_s

$F_s - W = ma$ [2] — m

Find m

W=mg [3]

Find k

object at rest

$F_{so} = -k(y_1 - y_o)$

$F_{so} = -ky_1$ [4] — F_{so}

Find F_{so}

$F_{so} - W = 0$ [5]

5 unknowns, 5 equations

From 5

$F_{so} = W$ into 4

$W = -ky_1$

$-\frac{W}{y_1} = k$ into 1

$F_s = \frac{W}{y_1} y_2$ [1]

From 3 into 2

$F_s - W = \frac{W}{g} a$

$F_s = \frac{W}{g} a + W$ into 1

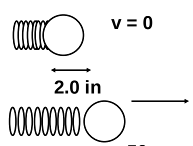
$W \frac{y_2}{y_1} = \frac{W}{g} a + W$

$\boxed{y_2 = (\frac{a}{g} + 1) y_1}$ independent of mass of object

$1.1(-6in) = y_2$

$\boxed{y_2 = -6.6 \text{ in}}$

You are asked to choose replacement springs for a pinball machine. The spring is used to launch a small 50 gram steel ball to begin the game. In order for the game to be fun, the ball should leave the spring at a speed of 10 ft/sec. At the beginning of the game, the ball is at rest at the end of a spring which has been compressed 2.0 inches from its unstretched length. When you release the spring, it launches the ball horizontally. Assuming that friction can be neglected, what should be the spring constant of the spring you choose?



Question: What is spring constant?

Approach:

Use spring force to relate the spring constant to the spring's force on the ball

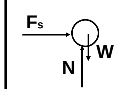
Relate that force to the acceleration of the ball using Newton's 2nd law.

Use the definition of acceleration to relate the acceleration of the ball to it's change of velocity.

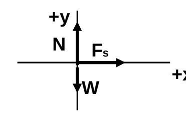
Will I need the definition of velocity also?

Neglect friction.

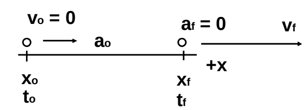
Free body diagram of the ball



Force diagram of the ball



Motion diagram of the ball



$x_o = -2.0$ in — $x_f = 0$ — m = 50 gr

$t_o = 0$ — $t_o = ?$

$v_o = 0$ — $v_f = 10$ ft/s

$a_o = ?$ — $a_f = 0$

Target quantity: k

Quantitative relationships:

$F_s = -kx$ $\quad a_x = \frac{dv_x}{dt}$ $\quad v_x = \frac{dx}{dt}$

$\Sigma F_x = ma_x$

PLAN — unknowns

Find k — k

$F_s = -kx$ [1] — F_s

Find F_s

From force diagram, forces in x direction

$F_s = ma_x$ [2] — a_x

Find a_x

$a_x = \frac{dv_x}{dt}$ [3] — t

Find t

$v_x = \frac{dx}{dt}$ [4]

---

[4] $v_x = \frac{dx}{dt}$

$dt = \frac{dx}{v_x}$ into [3]

$a_x = \frac{dv_x}{dt}$

$a_x = \frac{dv_x}{\frac{dx}{v_x}}$

$a_x = v_x \frac{dv_x}{dx}$ into [2]

$F_s = mv_x \frac{dv_x}{dx}$ into [1]

$-kx = mv_x \frac{dv_x}{dx}$

$-kx\,dx = mv_x dv_x$

$m\int v_x dv_x = -k\int x\,dx$

$\frac{1}{2} mv_x^2 = -\frac{1}{2} kx^2 + c$

Find c

at $x = x_o$, $v_x = 0$

$0 = -\frac{1}{2} kx_o^2 + c$

$c = \frac{1}{2} kx_o^2$

$\frac{1}{2} mv_x^2 = -\frac{1}{2} kx^2 + \frac{1}{2} kx_o^2$

$mv_x^2 = k(-x^2 + x_o^2)$

$\frac{mv_x^2}{-x^2 + x_o^2} = k$

Evaluate at x_f

$\frac{mv_f^2}{-x_f^2 + x_o^2} = k$

$\boxed{\frac{mv_f^2}{x_o^2} = k}$

$\frac{(50\ gr)(10\ ft/s)^2}{(-2\ in)^2} = k$

$\frac{5000\ (gr)(ft/s)^2}{4(in)^2} = k$

$\frac{5000\ (gr)(ft/s)^2 (\frac{12\ in}{ft})^2}{4(in)^2} = k$

$\boxed{1.8 x 10^5\ gr/s^2 = k}$

Are units of k correct?

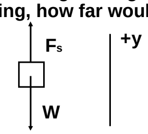
[F] = [k] [x]

[mass] [m/s] = [k] [m]

[mass/s] = [k] — ok

Is this reasonable?

If you hung a 100 gr object on this spring, how far would it stretch



0 = ky - W

mg = ky

g (m/k) = y

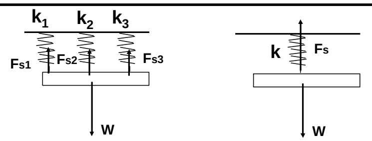
$(10\ m/s^2)(100\ gr)/(2x10^5\ gr/s^2) = y$

$(1/2) \times 10^{-2}$ m = y

(1/2) cm = y — a reasonable length

Example

Your team has just completed an inexpensive prototype of an inertial guidance system for use in cars. While building the prototype, your colleagues used three small springs to hold a part which is hanging vertically attached only to the springs.
The three springs have the same length and each have one end attached to a rigid bar and the other end attached to the part. Precise adjustments have been made to the motion of the part by using a different spring constant for each spring. These spring constants are given in the design report. Now to make the final design less expensive and more reliable, your manager tells you to replace the three springs with a single spring, with the specifications you are to determine, without changing the design of the system.



Find the spring constant of a single spring which has the same behavior as the 3 springs

Use spring force law

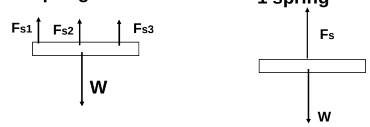
Single spring should have the same displacement and acceleration as the 3 springs when the object is hanging on it

Use relationship between force and accel

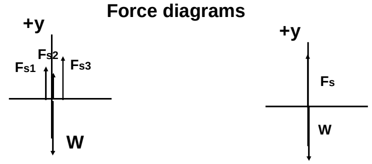
Use Newton’s 2nd law

Free body diagrams

3 springs   1 spring



Force diagrams



Given : k_1, k_2, k_3 , W

Target quantity: k

Use

$\sum F_y = ma_y$ $F_s = ky$

Plan		unknowns
Find k		k
single spring		
$F_s = ky$	1	F_s , y
Find F_s		
$\sum F_y = F_s - W$	2	ΣF_Y
Find ΣF_Y		
$\sum F_y = ma_y$	3	m, a_y
Find m		
$W = mg$	4	
Find a_y		
three springs		
$\sum F'_y = ma_y$	5	$\Sigma F'_Y$
Find $\Sigma F'_Y$		
$\sum F'_y = F_{s1} + F_{s2} + F_{s3} - W$	6	F_{s1} , F_{s2} , F_{s3}
Find F_{s1} , F_{s2} , F_{s3}		
$F_{s1} = k_1y$	7	
$F_{s2} = k_2y$	8	
$F_{s3} = k_3y$	9	

Is the problem solved?

10 unknowns, 9 equations

Which unknown is missing?

Do we know anything else useful?

Can’t think of anything.

Try for a solution anyway

Will any unknowns (especially y) cancel ?

Check plan

Put 7, 8, 9 into 6

$\sum F'_y = k_1y + k_2y + k_3y - W$ into 5

$k_1y + k_2y + k_3y - W = ma_y$ into 3

$\sum F_y = k_1y + k_2y + k_3y - W$ into 2

$F_s - W = k_1y + k_2y + k_3y - W$

$F_s = k_1y + k_2y + k_3y$ into 1

$ky = k_1y + k_2y + k_3y$ yes! y cancels out

$\boxed{k = k_1 + k_2 + k_3}$

Another plan based on another approach

The only influence on the motion of the object

Forces exerted by other objects

Earth, Springs

If the sum of the forces on the object is the same for 3 springs and 1 spring the motion will be the same

Plan		unknowns
Find k		k
single spring		
$F_s = ky$	1	F_s , y
Find F_s		
$\sum F_y = F_s - W$	2	ΣF_Y
Find ΣF_Y		
$\sum F_y = F_{s1} + F_{s2} + F_{s3} - W$	3	F_{s1} , F_{s2} , F_{s3}
Find F_{s1} , F_{s2} , F_{s3}		
$F_{s1} = k_1y$	4	
$F_{s2} = k_2y$	5	
$F_{s3} = k_3y$	6	

Put 4, 5, 6 into 3

$\sum F_y = k_1y + k_2y + k_3y - W$ into 2

$F_s - W = k_1y + k_2y + k_3y - W$

$F_s = k_1y + k_2y + k_3y$ into 1

$ky = k_1y + k_2y + k_3y$ y cancels out

$\boxed{k = k_1 + k_2 + k_3}$