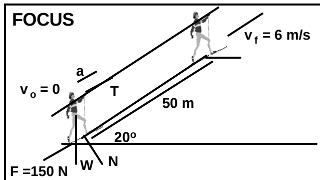
Using Potential Energy

You have a job providing the engineering help for an architect in Colorado. You are currently designing a cable tow to pull skiers up a hill so they can ski down. The customer would like the cable tow to pull a skier uphill at constant acceleration from the bottom reaching a speed of 6 m/s at the top. You need to determine what type of cable you should purchase. The hill is 50 m long and inclined at 20 degrees from the horizontal. By measuring skier speeds on a downhill run, you know there is a friction force of 150 N between the skis and the snow independent of the skier's weight.

FOCUS



Question: What is the force on the cable?

Approach:

Use conservation of energy to relate the final speed and the forces.

Include the Earth in the system and use gravitational potential energy

Initial time: just after grabs the rope at bottom
Final time: just as skier gets to the top

The rope force (T), and the frictional force (f) have components along the displacement.

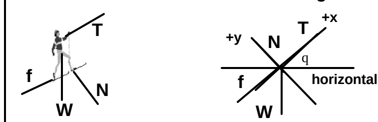
Energy transfer occurs

Choose one axis along the ski slope

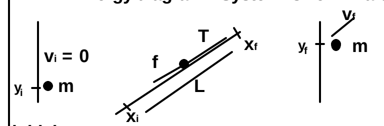
Might need to estimate mass of skier

PHYSICS DESCRIPTION

Free body diagram / Force diagram

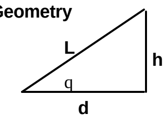


Energy diagram — System: skier + Earth



y_i = 0
q = 20°
v_f = 6 m/s
v_i = 0
L = 50 m
f = 150 N

Geometry



Target quantity: T

Quantitative relationships:

Conservation of Energy

$E_f - E_i = E_{input} - E_{output}$

System energy is kinetic + potential

$E_i = KE_i + GPE_i = 0 + 0$

$E_f = KE_f + GPE_f = \frac{1}{2} m v_f^2 + mgh$

Energy transferred to and from system

$E_{input} = \left|\int_0^L \vec{T} \cdot d\vec{\ell}\right| = \left|\int_0^L T d\ell\right| = \left|T \int_0^L d\ell\right| = TL$

$E_{output} = \left|\int_0^L \vec{f} \cdot d\vec{\ell}\right| = \left|\int_0^L f d\ell\right| = \left|f \int_0^L d\ell\right| = fL$

Conservation of Energy

$$\frac{1}{2} m v_f^2 + mgh - 0 = TL - fL$$

Plan: — unknowns

Find T — T

Conservation of energy

$$\frac{1}{2} m v_f^2 + mgh - 0 = TL - fL$$ — h, m

Find h

$$\sin q = \frac{h}{L}$$

3 unknowns, 2 equations

Need to estimate the skier's mass if it does not cancel out.

$$\frac{1}{2} m v_f^2 + mgL\sin q = TL - fL$$

$$\frac{1}{2L} m v_f^2 + mg\sin q + f = T$$

Mass of skier does not cancel

Estimate it at 100 kg

$$\frac{1}{2L} m v_f^2 + mg\sin q + f = T$$

Check units

[T] = [force] , [f] = [force] , [mg sin q] = [force]

$[\frac{1}{2L} m v_f^2] = [\text{mass}] \frac{[\text{m}^2 / \text{s}^2]}{[\text{m}]}$

= [mass] [acceleration]
= [force]

ok all units are force units

$T = \frac{1}{2\,(50\text{ m})}(100\text{ kg})(6\text{ m/s})^2$

$+ (100\text{ kg})(9.8\text{ m/s}^2) \sin 20° + (150\text{ N})$

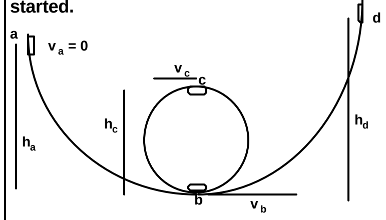
36 N + 335 N + 150 N = T

T = 521 N

Same as before

Example

A friend's child is playing with a toy cars and you decide to help by building a loop-the-loop track. You start a car on the entry track above the highest point on the circular part of the track. The car goes down the entry track around the circle and up an exit track. Based on the starting height of the car, you decide to calculate the speed of the car where it enters the circular part of the track as well as at the top of the circular part of the track and on the exit track 2 cm above where you started.



Use conservation of energy.

system: object + Earth

$$E_f - E_i = \Delta E_{transfer}$$

$E_i = KE_i + PE_i$ $\quad$ $E_f = KE_f + PE_f$

$\Delta E_{transfer} = 0$ $\quad$ Assume friction and air resistance not important

From a to b

Initial time at top of entrance ramp.

Assume the car starts from rest.

$$E_i = mgh_a$$

Final time at bottom of circle.

$$E_f = \frac{1}{2}mv_b^2$$

$$\frac{1}{2}mv_b^2 - mgh_a = 0$$

$$\boxed{v_b = \sqrt{2gh_a}}$$

From a to c

$E_i = mgh_a$ $\quad$ $E_f = \frac{1}{2}mv_c^2 + mgh_c$

$$\frac{1}{2}mv_c^2 + mgh_c - mgh_a = 0$$

$$\frac{1}{2}v_c^2 = gh_a - gh_c$$

$$\boxed{v_c = \sqrt{2g(h_a - h_c)}}$$

From a to d

$E_i = mgh_a$ $\quad$ $E_f = \frac{1}{2}mv_d^2 + mgh_d$

$$\frac{1}{2}mv_d^2 + mgh_d - mgh_a = 0$$

$$\frac{1}{2}v_d^2 = gh_a - gh_d$$

$$\boxed{v_d = \sqrt{2g(h_a - h_d)}}$$

$h_d > h_a$ not possible

Maximum height when KE = 0

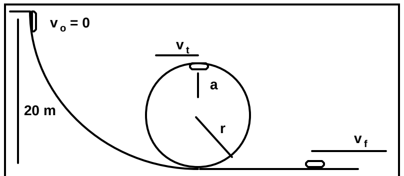
All energy is PE

Maximum height is initial height

if initial KE = 0 $\quad$ No energy input

Example

Your company has been hired to design a stunt for a new ice show. The star of the show enters by riding on a sled which starts from rest at the top of a curved ice track 20 m above the surface of the ice rink. The track leads down to the rink and, at that point, becomes a vertical circle which returns again to the rink. Your job is to calculate the maximum radius of the circle so that this daring loop-the-loop can be done without injuring the high priced star. Assume that you can neglect friction and air resistance as a first approximation.



What is the largest radius such that circular motion is possible?

$$a = \frac{v^2}{r}$$

Maximum radius means minimum acceleration

Acceleration cannot be smaller than g at top of circle

Get acceleration from forces on sled.

Get velocity from conservation of energy

System: sled + Earth

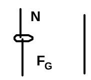
Gravitational potential energy

Initial time: start at top of track.

Final time: at top of circle

Ignore friction, air resistance

Free-body Diagram of sled at top of circle



$$\sum F_y = ma_y$$

$$\sum F_y = -F_G - N$$

minimum acceleration means minimum force

$$\sum F_y(\text{minimum}) = F_G$$

$$-mg = -ma$$

a(minimum) = g

Initial Energy $\quad$ Final Energy

y_o = 20 m $\quad$ $v_o = 0$

$y_f = 0$ $\quad$ m = ?

$y_t = 2r$

v_t = ?

$$E_i = KE_i + PE_i = mgy_o$$

$$E_f = KE_f + PE_f = \frac{1}{2}mv_t^2 + mgy_t$$

Conservation of Energy:

$$E_f - E_i = E_{in} - E_{out}$$

$$\frac{1}{2}mv_t^2 + mgy_t - mgy_o = 0$$

$$\frac{1}{2}v_t^2 + g(2r) - gy_o = 0$$

Target: r

Find	Equation		unknowns
			r
Find r	$\frac{1}{2}v_t^2 + g(2r) - gy_o = 0$	[1]	v_t
Find v_t	$a = \frac{v_t^2}{r}$	[2]	a
Find a	g = a	[3]	

3 unknowns, 3 equations ok

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[3] into [2] $g = \frac{v_t^2}{r}$

$rg = v_t^2$ Into [1]

$$\frac{1}{2}rg + g(2r) - gy_o = 0$$

$$\frac{1}{2}r + (2r) - y_o = 0$$

$$\frac{5}{2}r = y_o$$ units are ok both sides distance units

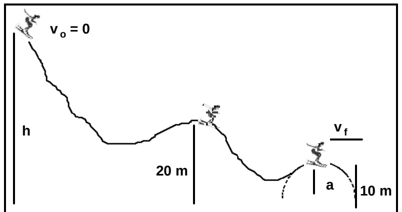
r = 8 m

The maximum height of the sled when travelling around the circle is 16 m, less than the initial height of 20 m.

This is reasonable since some of the initial potential energy has become kinetic energy at the top of the circle.

Example

A skier starts from rest on the slope on a summit and then skis over two successively lower hills of height 20 m and 10 m. The lowest hill is essentially a semi-circle centered at 0 height. The skier wants to leave the lowest hill at its top and fly through the air and asks you how far up the slope to start gliding down the hill. Assume friction and air resistance are negligible.



What is the initial height of the skier to leave the second hill at its top?

Skier stays on hill if the forces on the skier give the acceleration necessary to go in a circle.

$$a = \frac{v^2}{r}$$

Get the speed on the top of the hill from conservation of energy.

System: skier + Earth

Initial time: start on slope

Final time : top of 2nd hill

Get the necessary acceleration from forces

Free-body Diagram of skier on top of last hill

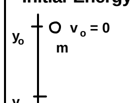
N - W = - ma

For the skier to follow the circular hill $a = \frac{v^2}{r}$

If skier leaves the hill

N = 0 - mg = - ma

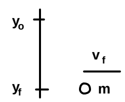
Initial Energy



$y_o = ?$ $v_o = 0$

m = ?

Final Energy



$y_f = 0$ $v_f = ?$

$E_i = KE_i + PE_i = mgy_o$ $E_f = KE_f + PE_f = \frac{1}{2}mv_f^2$

$\Delta E_{transfer} = 0$

Conservation of Energy $E_f - E_i = \Delta E_{transfer}$

$$\frac{1}{2}mv_f^2 - mgy_o = 0$$

target: y_o

Find	Equation		unknowns
Find y_o	$\frac{1}{2}v_f^2 - gy_o = 0$	[1]	y_o
			v_f
Find v_f	$a = \frac{v_f^2}{r}$	[2]	a
Find a	g = a	[3]	

3 unknowns, 3 equations

[3] into [2] $g = \frac{v_f^2}{r}$

$rg = v_f^2$ into [1]

$$\frac{1}{2}rg - gy_o = 0$$

$$\frac{1}{2}r = y_o$$ correct units both sides are distances

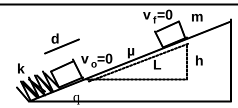
y_o = 5 m above the top of the hill

But that will not get you over the 1st hill

Need to be at a height of at least 20 m up the slope

Example

From your laboratory experience, you know it is difficult to measure the coefficient of kinetic friction between two surfaces. One of your lab partners suggest using a spring to propel a block up a ramp inclined at an angle from the horizontal that you measure. The block is to be held against a spring, compressing the spring a distance from its relaxed position that you measure. When the block is released, the spring expands and pushes the block upward along the ramp. The block leaves the spring, going a distance up the incline that you also measure? You can also measure the mass of the block and the spring constant. Will this procedure give you what you want?

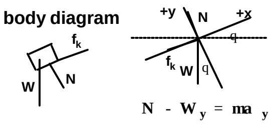


What is the coefficient of kinetic friction?

Use conservation of energy:
- system: block, Earth, and spring
  - Potential Energy: spring, gravitational
  - Energy transfer: friction
- Initial time: just after release from compressed spring
- Final time: just when block stops

Get frictional force from dynamics

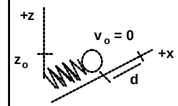
Free body diagram



$$N - W_y = ma_y = 0$$

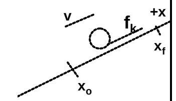
$$N - mg\cos\theta = 0$$

Initial Time



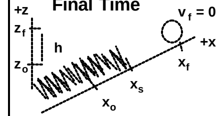
$$E_i = KE_i + PE_i = \frac{1}{2}kd^2$$

Energy Transfer



$$E_{out} = \int_0^L f_k\,dx$$

Final Time



$$E_f = KE_f + PE_f = mgh$$

Conservation of Energy: $E_f - E_i = E_{in} - E_{out}$

$$mgh - \frac{1}{2}kd^2 = -\int_0^L f_k\,dx$$

$$mgh - \frac{1}{2}kd^2 = -f_k L$$

target: μ_k

Plan

unknowns

Find μ_k — μ_k

$f_k = \mu_k N$ [1] — f_k, N

Find N

$N - mg\cos\theta = 0$ [2]

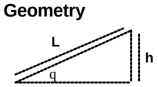
Find f_k

$mgh - \frac{1}{2}kd^2 = -f_k L$ [3] — h

Find h

$\sin\theta = \frac{h}{L}$ [4] 4 unknowns, 4 equations

Geometry



Execute the plan from the bottom up.

Put [4] into [3] and solve for f_k

$$mgL\sin\theta - \frac{1}{2}kd^2 = -f_k L$$

$$-mg\sin\theta + \frac{1}{2L}kd^2 = f_k$$

Put into [1] along with [2] and solve for μ_k

$$-mg\sin\theta + \frac{1}{2L}kd^2 = \mu_k mg\cos\theta$$

$$\frac{-mg\sin\theta + \frac{1}{2L}kd^2}{mg\cos\theta} = \mu_k$$

check units

$$\frac{[force] + \frac{1}{2[distance]}[force \cdot distance]}{[force]} = [\mu_k]$$

Correct, μ_k has no units

How to Solve Problems Using Energy

1. Picture the situation
   - What is the system?
   - How is it moving?
     - Is there energy transfer?
     - What path does the object travel?
     - Is there potential energy?

   Carefully identify the initial time and the final time you want to consider.

   Can you account for all of the energy of your system at those times?

   KE + PE

   Can you account for all energy transfers between those times?

$$E_{transfer} = \int_{\ell_o}^{\ell_f} \vec{F} \cdot d\vec{\ell}$$

2. Define your quantities with respect to a coordinate system.

Make sure you know which direction is + and which is -.

Force, position

Use your defined quantities to write down the conservation of energy equation for your system.

Keep track of the signs.

Keep track of the target quantity.

Do you need to know anything else in addition to conservation of energy?

Force laws

Kinematics

3. Identify all unknowns in your conservation of energy equation and relate them by equations to other information or principles physics.

4. Solve the system of equations to get your target quantity.

5. Check you answer

Correct units?

Reasonable behavior or value?

Did you answer the question?