Another 2-D Motion

Uniform Circular Motion

Qualitative Aspects

Quantitative Relationships

From definitions
velocity
acceleration

Vector Components

Some jargon

Period

Angular speed

Uniform Circular Motion

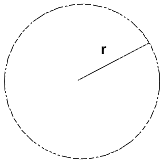
- Trajectory--circle
- Speed--constant

What is the velocity and acceleration of an object in uniform circular motion ?

What do we know about a circle

- Every point on the circle is the same distance from the center.

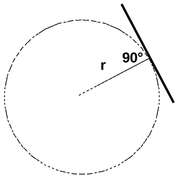
radius r



- Distance around the circle is 2πr

circumference C

- The radius is perpendicular to any tangent to the circle



- The motion repeats (periodic)

Time to make one full revolution
period T

- Average speed for one period

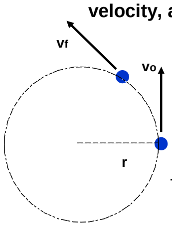
$\frac{C}{T}$

- Average velocity for one period

0

- Instantaneous speed
- Magnitude of instantaneous velocity

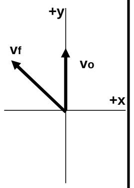
Describe the motion of an object:
velocity, acceleration



$v_o = v_f$

BUT

$\vec{v}_o$ NOT = $\vec{v}_f$



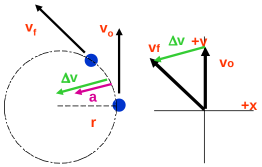
Velocity
- Same magnitude
- Changes direction

Acceleration
- Not zero
- Changes direction

Both velocity and acceleration depend on where you are on the circle

Direction
Magnitude

Change of Velocity



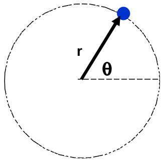
$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t}$$

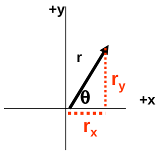
$$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$$

Direction of acceleration is direction of change of velocity

Quantitative

Describing a position





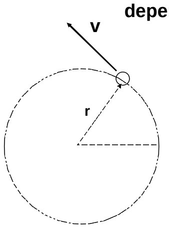
$$\frac{r_x}{r} = \cos\theta$$

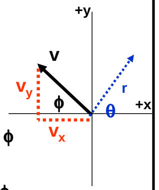
$$\frac{r_Y}{r} = \sin\theta$$

$$\vec{r} = r_x\hat{i} + r_y\hat{j}$$

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

How does the velocity of an object
depend on its position?





$$\frac{v_x}{v} = \cos\phi$$

$$\frac{v_y}{v} = \sin\phi$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$\vec{v} = -v\cos\phi\hat{i} + v\sin\phi\hat{j}$$

What does ϕ have to do with position?

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

θ + 90° + ϕ = 180°

ϕ = 90° - θ

$$\vec{v} = -v\cos\left(90^o - \theta\right)\hat{i} + v\sin\left(90^o - \theta\right)\hat{j}$$

$$\vec{v} = -v\sin\theta\hat{i} + v\cos\theta\hat{j}$$

$v_x = -v \sin\theta$ $\quad$ $v_y = v \cos\theta$

Velocity Magnitude

$$v^2 = v_x^2 + v_y^2$$

$$v_x = \frac{dx}{dt}$$

$$v_x = \frac{d(r\cos\theta)}{dt}$$

r is a constant

$$v_x = r\frac{d(\cos\theta)}{dt}$$

$$v_x = -r\sin\theta\frac{d\theta}{dt}$$

Do the same for v_y

$$v_y = \frac{dy}{dt}$$

$$v_y = \frac{d(r\sin\theta)}{dt}$$

$$v_y = r\cos\theta\frac{d\theta}{dt}$$

Should agree with what you got in lab

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v = \sqrt{r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2}$$

$$v = r\frac{d\theta}{dt}\sqrt{\sin^2\theta + \cos^2\theta} = r\frac{d\theta}{dt}$$

$$\boxed{\frac{v}{r} = \frac{d\theta}{dt}}$$

a constant

The rate that the angle changes is constant if the speed is constant

note:

Pythagorean Theorem

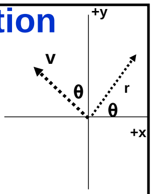
$$r^2 = x^2 + y^2$$

$$r^2 = r^2\cos\theta^2 + r^2\sin^2\theta$$

$$r^2 = r^2(\cos\theta^2 + \sin^2\theta)$$

$$1 = (\cos\theta^2 + \sin^2\theta)$$

Acceleration



$$a^2 = a_x^2 + a_y^2$$

$$a_x = \frac{dv_x}{dt}$$

$$a_y = \frac{dv_y}{dt}$$

velocity

$$v_x = -r\sin\theta\frac{d\theta}{dt}$$

$$\frac{v}{r} = \frac{d\theta}{dt}$$

$$v_y = r\cos\theta\frac{d\theta}{dt}$$

$$v_x = -r\sin\theta\frac{v}{r} = -v\sin\theta$$

$$v_y = r\cos\theta\frac{v}{r} = v\cos\theta$$

$$a_x = \frac{d}{dt}\left(-v\sin\theta\right) = -v\cos\theta\frac{d\theta}{dt} = -\frac{v^2}{r}\cos\theta$$

$$a_y = \frac{d}{dt}\left(v\cos\theta\right) = -v\sin\theta\frac{d\theta}{dt} = -\frac{v^2}{r}\sin\theta$$

$$a_x = -\frac{v^2}{r}\cos\theta \qquad a_y = -\frac{v^2}{r}\sin\theta$$

$$a = \sqrt{\frac{v^4}{r^2}\cos^2\theta + \frac{v^4}{r^2}\sin^2\theta} = \frac{v^2}{r}\sqrt{\cos^2\theta + \sin^2\theta}$$

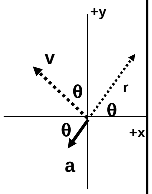
$$\boxed{a = \frac{v^2}{r}}$$

units of acceleration ?

$[\text{ m / s}]^2 / [\text{m}] = [\text{m / s}^2]$

correct units for an acceleratio

Direction of acceleration



$$\vec{a} = -a\cos\theta\hat{i} + -a\sin\theta\hat{j}$$

compare to

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

Direction of a is opposite to r

Replay

- Get acceleration into components

$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

- Use definition of accel. for each component

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt}$$

- Get definition of velocity for each component

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt}$$

- Use $\frac{d\theta}{dt} = \frac{v}{r}$
- Use Pythagorean Theorem

$$a^2 = a_x^2 + a_y^2$$

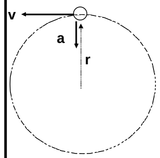
to get magnitude of acceleration

$$a = \frac{v^2}{r}$$

- Use components of acceleration to show

Acceleration is directed inwards
toward the center of the circle

Uniform Circular Motion



Any object following a circular path at a constant speed is accelerating.

- Direction: along radius toward center of circle.
- Magnitude: $a = v^2 / r$
- The acceleration is always perpendicular to the velocity

The SUM of the forces on the object must be towards the center of the circle

The distance once around a circle is its circumference (C)

- $C = 2 \pi r$

The time to go once around a circle is the period (T).

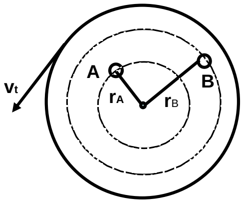
If the speed of the object is constant,

- $v = C / T$

Angular speed (rate angle changes) is $\frac{d\theta}{dt} = \frac{v}{r}$

Example

You have a job at a business which designs parts for jet engines. Your task, as a member of a safety team is to compare the motion of two different parts located on a disk attached to the turbine shaft. The part furthest from the turbine shaft is three times the distance from the shaft as the other part. The turbine shaft goes through the center of the disk and is perpendicular to it. Assume the disk is in uniform circular motion at n rotations/sec.



Looking down from above

$r_B = 3\ r_A$

n rotations per sec

Question: Compare v_A and v_B

Compare a_A and a_B

Approach:

Compare means make an equation involving the two quantities in question.

Both objects have a constant speed

Both objects take the same time to go around a full circle

Same period

The objects go around circles of different circumference.

The objects have different speeds

Use definition of average speed

For constant speed, instantaneous speed equals average speed

Use relationship between instantaneous speed and acceleration for uniform circular motion

Qualitatively, which is larger

v_A or v_B

a_A or a_B

Why?

Qualitative Analysis

Magnitude of instantaneous velocity

Circle B is larger than circle A

Object B takes same time as object A to go once around the circle

$v_B > v_A$

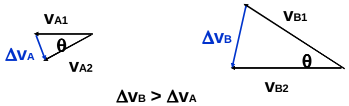
Magnitude of instantaneous acceleration

In any time interval,

the change of direction of A

is the same as

the change of direction of B



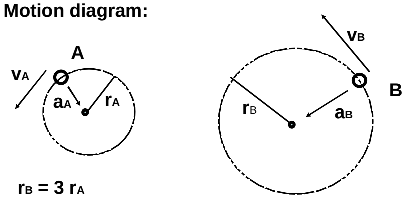
$\Delta v_B > \Delta v_A$

$\vec{a}_{avA} = \frac{\Delta \vec{v}_A}{\Delta t}$ $\vec{a}_{avB} = \frac{\Delta \vec{v}_B}{\Delta t}$

Δt is same for both objects

$a_{av\ B} > a_{av\ A}$

Motion diagram:



$r_B = 3\ r_A$

$T_B = T_A = T = \frac{1}{n}$ sec

Target quantities:

$v_A = f(v_B)$

$a_A = f(a_B)$

Quantitative relationships:

$s_{av} = \frac{\text{distance}}{\Delta t}$

$s_{av} = v$ for constant speed

$a = \frac{v^2}{r}$ for uniform circular motion

$C = 2\pi r$

Plan:

Find v_A

$v_A = \frac{C_A}{T}$ $\boxed{1}$

Find C_A

$C_A = 2\ \pi\ r_A$ $\boxed{2}$

Find r_A

$r_B = 3\ r_A$ $\boxed{3}$

Find r_B

$C_B = 2\ \pi\ r_B$ $\boxed{4}$

Find C_B

$v_B = \frac{C_B}{T}$ $\boxed{5}$

5 unknows, 5 equations

unknowns

v_A

C_A

r_A

r_B

C_B

$\boxed{5}$ $v_B = \frac{C_B}{T}$

$v_B T = C_B$ into $\boxed{4}$

$v_B T = 2\pi r_B$

$\frac{v_B T}{2\pi} = r_B$ into $\boxed{3}$

$\frac{v_B T}{2\pi} = 3r_A$

$\frac{v_B T}{6\pi} = r_A$ into $\boxed{2}$

$C_A = 2\pi \frac{v_B T}{6\pi}$

$C_A = \frac{v_B T}{3}$ into $\boxed{1}$

$v_A = \frac{\frac{v_B T}{3}}{T}$

$\boxed{v_A = \frac{v_B}{3}}$

Plan to find acceleration

unknowns

Find a_A — a_A

$a_A = \frac{v_A^2}{r_A}$ $\boxed{1}$ — v_A , r_A

Find v_A

from 1st part

$v_A = \frac{1}{3} v_B$ $\boxed{2}$

Find r_A

$r_A = \frac{1}{3} r_B$ $\boxed{3}$ — r_B

Find r_B

$a_B = \frac{v_B^2}{r_B}$ $\boxed{4}$

4 unknowns , 4 equations

$\boxed{4}$ $a_B = \frac{v_B^2}{r_B}$

$r_B = \frac{v_B^2}{a_B}$ into $\boxed{3}$

$r_A = \frac{v_B^2}{3a_B}$ into $\boxed{1}$

$a_A = \frac{v_A^2}{\frac{v_B^2}{3a_B}}$

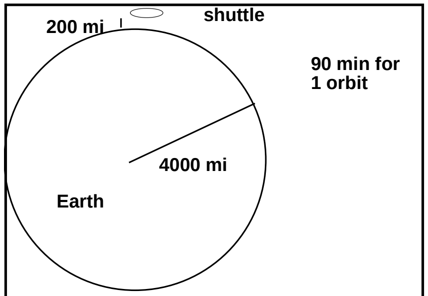
$\boxed{2}$ $v_A = \frac{1}{3} v_B$ into $\boxed{1}$

$a_A = \frac{(\frac{v_B}{3})^2}{\frac{v_B^2}{3a_B}}$

$\boxed{a_A = \frac{1}{3} a_B}$

Example

A space shuttle typically has a circular orbit around the earth at a height of 200 miles. It travels with a constant speed and completes one orbit in 90 minutes. The radius of the Earth is about 4000 miles. What is its acceleration?



Question: What is acceleration of shuttle?

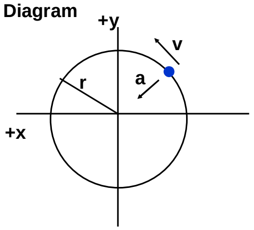
Approach:

Shuttle travels in a circle around the Earth

Uniform circular motion

Use expression for centripetal accel.

Diagram



r = 4200 mi

T = 90 min

Target quantity: a

Quantitative relationships:

$a = \frac{v^2}{r}$

$s = \frac{\text{distance}}{\Delta t}$

$C = 2\pi r$

v = s for constant s

PLAN:

unknowns

Find a

a

$$a = \frac{v^2}{r} \quad \boxed{1}$$

v

Find v

$$v = \frac{2\pi r}{T} \quad \boxed{2}$$

2 unknowns, 2 equations

$$a = \frac{\left(\frac{2\pi r}{T}\right)^2}{r}$$

$$\boxed{a = \frac{4\pi^2 r}{T^2}}$$

check units

$$a = \frac{4\pi^2(4200\ \text{mi})}{(90\ \text{min})^2}$$

$$\boxed{a = 20.5\ \frac{\text{mi}}{\text{min}^2}}$$

Evaluate:

$$\frac{[\text{distance}]}{[\text{time}]^2}$$

Are correct units for accel.

The question is answered since the acceleration of the shuttle is calculated

Does a make sense? Compare it to g.

$$a = 20.5\ \frac{\text{mi}}{\text{min}^2}\ \left(\frac{1\ \text{min}}{60\ \text{sec}}\right)^2\ \left(\frac{5280\ \text{ft}}{1\ \text{mi}}\right)$$

$$a = 30\ \frac{\text{ft}}{\text{sec}^2}$$

slightly less than the acceleration if you drop it on the surface of the Earth

This makes sense since 200 miles is very close to the surface of the Earth compared to 4000 miles.